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Electricity markets: Designing auctions where suppliers have uncertain costs.*

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Abstract

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1 Introduction

We analyse multi-unit auctions where producers submit bids before demand, production capacities and production costs are fully known. Our model accounts for asymmetric information in suppliers' production costs and considers unexpected outages and intermittent output, such as those from renewable energy sources. Our analysis is, for example, of relevance for European wholesale electricity markets, where the European Commission is introducing regulations on submission and publication of data to increase the market transparency (EU, 2011;2013). We are interested in how such regulations and the auction format (uniform or discriminatory pricing) influence the competitiveness of the resulting market outcomes. Our results also suggest that the bidding format is of importance for market performance.

In electricity markets, the production costs can, to some extent, be estimated from engineering data of plants and fuel price indexes. Still, a generation unit owner has private information about the specific price paid for its quality of fuel and how a plant is operated in detail. We believe that the cost uncertainty is largest in hydro-dominated markets. The opportunity cost of using water stored in a reservoir is estimated from a discounted prognosis of future electricity prices. The uncertainty in the opportunity cost is to a large extent common for hydro-power producers, but the uncertainty also has private components such as a producer's internal risk-adjusted discount rates. If water is scarce and inflows are small, then producers have the possibility to store water for a long period of time, sometimes several years. In this case, the opportunity cost depends on a prediction of electricity prices several years later, so the uncertainty becomes large. The uncertainty in the opportunity cost is further exacerbated by political risks, the possibility of regulatory intervention and producers' subjective probabilities of such events during the planned storage period.

We consider a multi-unit auction with two producers, where an auctioneer must buy from both producers when demand is high. Accepted bids are either paid a uniform or a discriminatory price. In uniform-price procurement auctions, all accepted bids are paid a uniform clearing price, which is set by the highest accepted bid (Krishna, 2010). In a discriminatory auction, all accepted bids are paid their own bid price (Krishna, 2010). The auctioneer's demand is uncertain and realized after bids have been submitted. Similar to von der Fehr and Harbord (1993), each firm makes a sealed bid where it offers its entire production capacity at one unit price in a one-shot game. In our case, the production capacity could be realized after bids have been submitted. We also generalize von der Fehr and Harbord (1993) by introducing uncertain interdependent costs. Analogous to Milgrom and Weber's (1982) auction model for single objects, each firm makes its own estimate of production costs based on private imperfect information that it receives, and then makes a bid.[1] As is customary in game theory, we refer to this private information as a private *signal*.

[1] Milgrom and Weber (1982) analyse an analogous sales auction, so in their setting each agent estimates the value of the good that the auctioneer is selling.

Similar to Milgrom and Weber (1982), we solve for a symmetric Bayesian NE and consider signals that are drawn from a non-parametric distribution. Bidding behaviour depends on the correlation or affiliation of signals. Affiliated signals are such that if the signal of one player increases, then it increases the probability that the competitor has a high signal relative to the probability that the competitor has a low signal.

As in von der Fehr and Harbord (1993), our results depend on whether producers are pivotal or not. A producer is pivotal if its competitors do not have enough production capacity to meet the realized demand. Unlike von der Fehr and Harbord (1993), market prices are higher when producers are pivotal with a larger margin in our setting. There is one exception in our analysis of the uniform-price auction, where the market price could decrease when demand increases at the critical point where producers switch from being non-pivotal to being pivotal. The reason for this counter-intuitive result is that the price-setting bid switches from the lowest bid to the highest bid at this critical point, which drastically changes the bidding behaviour. Bidders are always non-pivotal in single object auctions with at least two participants, while the number of pivotal producers in wholesale electricity markets depends on the season and the time-of-day (Genc and Reynolds, 2011), but also on market shocks. Pivotal supplier indicators are used in practice to predict market power in electricity markets.

We show that the auctioneer prefers a design with uniform pricing to discriminatory-pricing when signals are sufficiently close to being independent. The advantages of the uniform-price auction increase when cost uncertainty increases and if bidders are pivotal with a higher probability. This supports the widespread use of uniform-pricing in wholesale electricity markets.[2]

We find that mark-ups decrease if producers receive similar information, i.e. signals are correlated. This is related to Vives (2011) who find that mark-ups would decrease for less noisy cost information. It is also known from previous work that disclosure of information improves competition in single object auctions (Milgrom and Weber, 1982). Taken together, these results suggest that publicly available information of relevance for production costs – such as weather conditions, fuel prices, prices of emission permits – improves the market competitiveness. It is also easier for a producer to estimate the production costs of its competitors if the market operator discloses detailed historical bid data. Thus, our results indicate that the transparency increasing measures of the European Commission should improve the performance of European electricity markets. In addition, information from financial markets just ahead of the physical markets would lower the market uncertainty. Trading of long-term contracts helps producers predict future electricity prices, which lowers the uncertainty in the opportunity cost of water. Our results also suggest that political risks are harmful for competition in hydro dominated markets, especially when water is scarce. Thus,

[2] Most deregulated wholesale electricity markets use uniform-pricing. Two exceptions are Britain and Iran, which use discriminatory pricing. In addition, some special auctions in the electricity market, such as counter-trading in the balancing market and/or the procurement of power reserves, sometimes use discriminatory pricing (Holmberg and Lazarzcyk, 2015; Anderson et al., 2013).

we recommend clearly defined contingency plans for intervention by the market operator and the regulator under extreme system conditions. This could potentially mitigate the extraordinarily high-priced periods that typically accompany low water conditions in hydro dominated wholesale electricity markets such as California, Colombia, and New Zealand.

Increased transparency lowers the payoff of producers, which also suggests that producers would try to conceal their cost estimates from each other. This has similarities to Gal-Or (1986) who shows that producers that play a Bertrand equilibrium would try to conceal their costs from each other. According to our results, increased transparency would only be helpful up to a point, because there is a lower bound on equilibrium mark-ups when producers are pivotal. Another caveat is that we only consider a single shot game. As argued by von der Fehr (2013), there is a risk that increased transparency in European electricity markets facilitates tacit collusion in a repeated game.

We show that bids in a discriminatory auction are determined by the expected sales of the highest and lowest bidder, respectively. In our setting, the variance in these sales after bids have been submitted – due to uncertainties in demand, outages and intermittent renewable production – will not influence the bidding behaviour of producers in the discriminatory auction. Bidding in the uniform-price auction is also insensitive to variances, as long as the market shocks are not sufficiently large to occasionally change the pivotal status of at least one producer. If the pivotal status changes with a positive probability in a uniform-price auction, then we find for independent signals that mark-ups increase when there is a higher probability that producers are pivotal for given expected sales of the highest and lowest bidder.

Our results depend on the auction design, the demand side, production capacities, the information structure, expected costs and costs for the special case when information is symmetric. As long as those aspects are unchanged, our results do not depend on the extent to which the cost uncertainty is private, interdependent or common. This is quite different to Vives (2011). The reason is that producers in his setting choose linear supply functions and can therefore condition their output on every price. To a larger extent than in our model, his bidding format allows producers to condition their output on the competitor's information. If costs are common or positively interdependent, a producer therefore has an incentive to reduce output when the price is unexpectedly high (when the competitor has received a high cost signal) and increase the output when the price is unexpectedly low (when the competitor has received a low cost signal). This will make supply functions steeper or even downward sloping, which will significantly harm competition. If costs are common, mark-ups can be as high as for the collusive outcome (Vives, 2011). Thus, our results and the results in Vives (2011) indicate that when the cost uncertainty is common or strongly interdependent, which should often be the case in wholesale electricity markets, then it should be optimal to limit the number of allowed steps in producers' supply functions in order to give producers less freedom to condition their output on competitors' signals. Most wholesale electricity markets and other multi-units auctions have such constraints in the bidding format.

Our study focuses on procurement auctions, but the results are analogous for divisible-good sales auctions. Purchase constraints in sales auctions correspond to production capacities in our setting. As an example, U.S. treasury auctions have the 35% rule, which prevents a single bidder from buying more than 35% of the sold securities. This is to avoid the outcome where a single bidder corners the market. On the other hand, the 35% rule increases the risk that a bidder will be pivotal. To some extent, bidders' financial constraints would also correspond to production capacities.[3] The supply of treasury bills, which corresponds to the auctioneer's demand in our model, is sometimes uncertain when bids are submitted due to an uncertain amount of non-competitive bids (Wang and Zender, 2002) or because the auctioneer wants to wait for the latest market news before announcing its supply of treasury bills. Our results also indicate that the uniform-price format is preferable. Thus, the U.S. Treasury should have gained from switching from the discriminatory format to the uniform-price format during the 1990s. Most other treasury auctions around the world use discriminatory pricing (Bartolini and Cottarelli, 1997). Our results also suggest that central banks could benefit from disclosing more information.

The paper is organized as follows. Section 2 compares our paper with the previous literature. Section 3 formally introduces our model, which is analysed for auctions with discriminatory and uniform-pricing in Section 4. The paper is concluded in Section 5. All proofs are in the Appendix.

2 Comparison with previous studies

Our model is inspired by classical auction theory. In the special case where producers are non-pivotal with certainty, our model corresponds to the single object auction analysed by Milgrom and Weber (1982). Our main methodological contribution is that we generalize their model to the pivotal case, where competitors do not have enough production capacity to meet all of the auctioneer's demand. Our model also generalizes Parisio and Bosco (2003), which is restricted to producers with independent private costs in uniform-price auctions.

Divisible-good auctions often have restrictions on how many bids each producer can submit or, equivalently, how many steps a producer is allowed to have in its stepped supply function. In practice, a producer is normally allowed to make more than the single bid that is considered here and in von der Fehr and Harbord (1993). However, as shown by Genc (2009) and Anderson et al. (2013), there are many circumstances in discriminatory auctions where a producer finds it optimal to offer its whole production capacity at one price, and then the equilibrium is not influenced by the single-bid restriction. Similar to the extension of a single-bid model to a multi-bid model by Fabra et al. (2006), we could also argue that it is enough to solve for the marginal bids of the two producers as long

[3]Financial constraints of bidders partly explain the bidding behaviour in security auctions. They are, for example, analysed by Che and Gale (1998).

as other bids never clear the market, i.e. the uncertainties in the market are sufficiently small. Kastl (2012) and Wolak (2007) have developed empirical models where each firm submits multiple bids in markets where the market uncertainties could be large. Ausubel et al. (2014) extend Milgrom and Weber (1982) to the case with two bids per non-pivotal bidder. This complicates the ranking of auction formats, so that it becomes ambiguous. Holmberg et al. (2013) solve for equilibria of stepped supply functions in divisible-good auctions with discrete price levels. Supply functions with a large number of steps are often approximated by continuous supply functions (Klemperer and Meyer, 1989; Wilson, 1979).

Previous models of strategic bidding in wholesale electricity markets normally focus on demand uncertainties and tend to neglect cost uncertainties (Green and Newbery, 1992; Anderson and Hu, 2008; Holmberg and Newbery, 2010; von der Fehr and Harbord, 1993). Vives (2011) is one exception. He considers a bidding format that is different from ours, where producers compete with linear supply functions in a uniform-price auction. Signals are normally distributed in Vives (2011), while we consider a non-parametric distribution. Rostek and Weretka (2012) extend Vives' (2011) setting to a double auction.

In order to facilitate comparisons with previous studies where the costs are assumed to be common knowledge, we derive results for the limit where the cost uncertainty decreases until the costs are almost surely common knowledge. In this limit, our model of the discriminatory auction corresponds to the classical Bertrand game. We get the competitive outcome with zero mark-ups for this limit when non-pivotal producers have weakly affiliated signals, both for uniform-price and discriminatory auctions. This concurs with the competitive outcomes for non-pivotal producers in the Bertrand game, in von der Fehr and Harbord (1993) and in Fabra et al. (2006). If signals are independent and producers pivotal, it follows from Harsanyi's (1973) purification theorem that in the limit when costs are almost surely common knowledge, our Bayesian Nash equilibria correspond to the mixed-strategy NE that Anderson et al. (2013), Anwar (2006), Fabra et al. (2006), Genc (2009), Son et al. (2004) and von der Fehr and Harbord (1993) derive for uniform-price and discriminatory auctions.[4] Similar mixed strategy NE also occur in the Bertrand-Edgeworth game (Edgeworth, 1925; Allen and Hellwig, 1986; Beckmann, 1967; Levitan and Shubik, 1972; Maskin, 1986; Vives, 1986).[5]

As for example illustrated by Klemperer and Meyer (1989), there can be multiple NE in divisible-good auctions when some bids are never price-setting. This is not an issue in discriminatory auctions or in our general model of the uniform-price auction where the pivotal status of bidders changes with a positive probability. However, in the special case where producers are pivotal with certainty in a uniform-price auction, there is another equilibrium, in addition to the symmetric equilibrium that we calculate. The other equilibrium is asymmetric and was discovered by von der Fehr and Harbord (1993). In our discussion, we refer to this as

[4] Blázquez de Paz (2014) generalizes these mixed-strategy NE to consider transmission constraints.

[5] Deneckere and Kovenock (1996) and Osborne and Pitchik (1986) generalize the Bertrand-Edgeworth game to a setting with asymmetric costs and asymmetric production capacities, respectively.

the high-price equilibrium, because the price-setting producer bids at the highest possible price (the reservation price). The other producer just bids sufficiently low to deter deviations by the high-price bidder.

In our setting, uniform-pricing and discriminatory pricing are equivalent when firms are non-pivotal. Fabra et al. (2006) get the same result for costs that are common knowledge. However, our ranking differs from that of Fabra et al. (2006) when firms are pivotal with certainty. The reason is that they select the high-price equilibrium in their analysis of the uniform-price auction, which makes this auction format very unattractive for an auctioneer. Holmberg (2009) and Hästö and Holmberg (2006) identify circumstances where the auctioneer prefers the pay-as-bid auction to the uniform-price auction when suppliers have costs that are common knowledge and offer continuous supply functions. Pycia and Woodward (2015) use a similar model and show that pay-as-bid and uniform-price auctions are revenue equivalent if the auctioneer chooses the reservation price and its supply of goods optimally.

3 Model

There are two risk-neutral producers in the market. Each producer $i \in \{1, 2\}$ receives a private signal s_i with imperfect cost information. The joint probability density $\chi(s_i, s_j)$ is continuously differentiable and symmetric, so that $\chi(s_i, s_j) \equiv \chi(s_j, s_i)$. Moreover, $\chi(s_i, s_j) > 0$ for $(s_i, s_j) \in (\underline{s}, \overline{s}) \times (\underline{s}, \overline{s})$.[6] We say that signals are weakly affiliated when[7]

$$\frac{\chi(u, v')}{\chi(u, v)} \leq \frac{\chi(u', v')}{\chi(u', v)}, \tag{1}$$

where $v' \geq v$ and $u' \geq u$. Thus, if the signal of one player increases, then it (weakly) increases the probability that its competitor has a high signal relative to the probability that its competitor has a low signal. It can be shown that signals are weakly affiliated if and only if $\ln \chi(u, v)$ is supermodular (Krishna, 2010). We say that signals are weakly unaffiliated when the opposite is true, i.e.

$$\frac{\chi(u, v')}{\chi(u, v)} \geq \frac{\chi(u', v')}{\chi(u', v)}, \tag{2}$$

where $v' \geq v$ and $u' \geq u$. Note that independent signals are both weakly affiliated and weakly unaffiliated.

We let

$$F(s_i) = \int_{-\infty}^{s_i} \int_{-\infty}^{\infty} \chi(u, v)\, dv du$$

[6] We do not require $\chi(s_i, s_j) > 0$ at the boundary, but $\frac{\chi_1(u,\overline{s})}{\chi(u,\overline{s})} = \frac{\chi_2(\overline{s},u)}{\chi(\overline{s},u)}$ is assumed to be bounded for $u \in [\underline{s}, \overline{s}]$.

[7] Milgrom and Weber (1982) call such signals affiliated. We write *weakly* affiliated to stress when the condition is also satisfied for independent signals.

denote the marginal distribution, i.e. the unconditional probability that supplier i receives a signal below s_i. We let

$$f(s_i) = F'(s_i).$$

As in von der Fehr and Harbord (1993), we consider the case when each firm's marginal cost is constant up to its production capacity constraint $\widetilde{q}_i$. But in our setting, $\widetilde{q}_i$ and marginal costs are uncertain when bids are submitted. The production capacities of the two producers could be correlated, but they are symmetric information and we assume that they are independent of production costs and signals. Capacities are symmetric ex-ante, so that $\mathbb{E}[\widetilde{q}_i] = \mathbb{E}[\widetilde{q}_j]$. Realized production capacities are assumed to be observed by the auctioneer when the market is cleared.[8] Costs are asymmetric information. In practice, a firm would normally know more about its own cost than about the competitor's cost. Firms' costs are symmetric ex-ante, i.e. $c_i(s_i, s_j) = c_j(s_i, s_j)$. We assume that

$$\frac{\partial c_i(s_i, s_j)}{\partial s_i} > 0, \tag{3}$$

so that a firm's marginal cost increases with respect to its own signal. A firm's marginal cost is allowed to decrease somewhat with respect to the competitor's signal, but we require that:

$$\frac{dc_i(s, s)}{ds} \geq 0. \tag{4}$$

Thus, if both producers would by (coincidence) receive the same signal s, then a producer's marginal cost is increasing with respect to that same signal. The special case with independent signals and $\frac{\partial c_i(s_i,s_j)}{\partial s_j} = 0$ corresponds to the private independent cost assumption, which is used in the analysis by Parisio and Bosco (2003). Krishna (2010) is another special case of our model where $c_i(s_i, s_j) = c_j(s_i, s_j)$ corresponds to the common cost/value assumption that is often used in the auction literature.

In our analysis, we frequently refer to the limit where production costs are almost surely common knowledge. Formally, we define:

Definition 1 *Production costs are almost surely common knowledge when* $\frac{dc_i(s,s)}{ds} = 0$ *for* $s \in [\underline{s}, \overline{s})$.

As in von der Fehr and Harbord (1993), demand can be uncertain $D \in [\underline{D}, \overline{D}]$. It could be correlated with the production capacities, but demand is assumed to be independent of the production costs and signals. In addition, it is assumed that all outcomes are such that $0 \leq D \leq \widetilde{q}_i + \widetilde{q}_j$, so that there is always enough production capacity to meet the realized demand. As in von der Fehr and Harbord (1993), demand is inelastic up to a reservation price $\overline{p}$. Analogous to Milgrom and

[8] Alternatively, similar to the market design of the Australian wholesale market, producers could first choose bid prices and later adjust production capacities at those prices just before the market is cleared. Anyway, we assume that the reported production capacities are verifiable by the auctioneer, so that bidders cannot choose them strategically.

Weber (1982), we assume that the reservation price is set at the highest relevant marginal cost realization, i.e. $\overline{p} = c_i(\overline{s}, \overline{s})$ for $i \in \{1, 2\}$. This assumption can be motivated by the fact that an auctioneer would lower its procurement cost by lowering the reservation price whenever $\overline{p} > c_i(\overline{s}, \overline{s})$.

After firms have received their private signals, each firm submits a bid with one unit price for its whole capacity in a one-shot game. We let $p_i(s_i)$ be the chosen bid of firm $i \in \{1, 2\}$ when it observes the signal s_i. The auctioneer accepts bids in order to minimize its procurement cost. In a uniform-price auction, the highest accepted bid price sets the uniform market price for all accepted bids. In a discriminatory auction, each accepted bid is paid its individual bid price.

Similar to classical auction theory, we solve for symmetric Bayesian Nash equilibria with the following properties: (i) the chosen bid price of firm $i \in \{1, 2\}$ is a twice differentiable function of its signal s_i and (ii) the bid is strictly monotonic in the firm's signal, i.e. $p_i'(s_i) > 0$ for $s_i \in (\underline{s}, \overline{s})$. Thus, the inverse $p_i^{-1}(p)$ always exists in equilibrium. Strict monotonicity also implies that ties occur with measure zero. Hence, the rationing rule will not influence the expected profit of producers in the equilibria for which we solve.

Ex-post, we denote the winning (low bid) producer, which gets a high output, by subscript H. The losing (high bid) producer, which gets a low output, is denoted by the subscript L. Winning and losing producers have the following expected outputs:

$$q_H = \mathbb{E}\left[\min\left(\widetilde{q}_H, D\right)\right] \tag{5}$$

and

$$q_L = \mathbb{E}\left[\max\left(0, D - \widetilde{q}_H\right)\right]. \tag{6}$$

The payoff of each producer is given by its revenue minus its realized production cost.

4 Analysis

4.1 Discriminatory pricing

Each firm is paid as bid under discriminatory pricing. The demand uncertainty and production capacity uncertainties are independent of the cost uncertainties. Thus, the expected profit of firm i when receiving signal s_i is:

$$\begin{aligned}\pi_i(s_i) = &\left(p_i(s_i) - \mathbb{E}\left[c_i(s_i, s_j) \mid p_j \geq p_i\right]\right) \Pr\left(p_j \geq p_i \mid s_i\right) q_H \\ &+ \left(p_i(s_i) - \mathbb{E}\left[c_i(s_i, s_j) \mid p_j \leq p_i\right]\right) \left(1 - \Pr\left(p_j \geq p_i \mid s_i\right)\right) q_L.\end{aligned} \tag{7}$$

Lemma 1 *In markets with discriminatory pricing:*

$$\begin{aligned}\frac{\partial \pi_i(s_i)}{\partial p_i} = &\Pr\left(p_j \geq p_i \mid s_i\right) q_H + \left(1 - \Pr\left(p_j \geq p_i \mid s_i\right)\right) q_L \\ &+ \left(p_i - c_i\left(s_i, p_j^{-1}(p_i)\right)\right) \frac{\partial \Pr\left(p_j \geq p_i \mid s_i\right)}{\partial p_i} \left(q_H - q_L\right).\end{aligned} \tag{8}$$

A producer i does not know the signal of its competitor j. However, when solving for the locally optimal bid, producer i is only interested in cases where the competitor, producer j, is bidding really close to p_i, which corresponds to the competitor receiving the signal $p_j^{-1}(p_i)$. This explains why $c_i\left(s_i, p_j^{-1}(p_i)\right)$ is the relevant cost of producer i in the above first-order condition. The first two terms on the right-hand side of (8) correspond to the price effect. This is what the producer would gain in expectation from increasing its bid by one unit if the acceptance probabilities were to remain unchanged. However, on the margin, a higher bid price lowers the probability of being the winning bidder by $\frac{\partial \Pr(p_j \geq p_i | s_i)}{\partial p_i}$. Switching from being the winning to the losing bidder reduces the accepted quantity by $q_H - q_L$. We refer to $\frac{\partial \Pr(p_j \geq p_i | s_i)}{\partial p_i}(q_H - q_L)$ as the quantity effect, the quantity that is lost on the margin from a marginal price increase. The mark-up for lost sales, $p_i - c_i\left(s_i, p_j^{-1}(p_i)\right)$, times the quantity effect gives the lost value of the quantity effect. This is the last term on the right-hand side of (8).

We solve for a symmetric Bayesian NE and henceforth, we drop firm-specific subscripts when appropriate. Producers may receive different signals but, in equilibrium, they react in the same way to an observed signal s as implied by the function $p(s)$. In equilibrium, the price effect, i.e. $\Pr(p_j \geq p_i | s_i)\, q_H + (1 - \Pr(p_j \geq p_i | s_i))\, q_L$, is proportional to

$$\int_{s}^{\overline{s}} \chi(s, s_j)\, ds_j q_H + \int_{\underline{s}}^{s} \chi(s, s_j)\, ds_j q_L \tag{9}$$

and the quantity effect, i.e. $\frac{\partial \Pr(p_j \geq p_i | s_i)}{\partial p_i}(q_H - q_L)$, is proportional to

$$(q_H - q_L)\frac{\chi(s, s)}{p'(s)}, \tag{10}$$

where $\frac{\chi(s,s)}{p'(s)}$ represents the probability density in terms of bid prices. We find it useful to introduce the following exogenous function, which is proportional to the ratio of the quantity and price effect.

Definition 2

$$H^*(s) := \frac{\chi(s, s)(q_H - q_L)}{\int_{s}^{\overline{s}} \chi(s, s_j)\, ds_j q_H + \int_{\underline{s}}^{s} \chi(s, s_j)\, ds_j q_L}. \tag{11}$$

Equilibrium bids are chosen optimally for each signal. The implication is that the price effect equals the value of the quantity effect for each signal. Thus, as formally proven in the proof of Proposition 1, equilibrium bids can be determined from the following ordinary differential equation (ODE):

$$p'(s) - (p - c(s, s))\, H^*(s) = 0. \tag{12}$$

The solution to this ODE is presented below.

Proposition 1 *The symmetric Bayesian Nash equilibrium bid in a discriminatory auction is given by:*

$$p(s) = c(s,s) + \int_s^{\overline{s}} \frac{dc(v,v)}{dv} e^{-\int_s^v H^*(u)du} dv \tag{13}$$

if $\frac{d}{ds}\left(\frac{\int_x^{\overline{s}} \chi(s,s_j)ds_j q_H + q_L \int_{\underline{s}}^x \chi(s,s_j)ds_j}{\chi(s,x)}\right) \geq 0$. *The equilibrium exists for more general probability distributions when* $\frac{dc(v,v)}{dv} > 0$. *In the limit when costs are almost surely common knowledge, the symmetric Bayesian Nash equilibrium can be simplified to:*

$$p(s) = c(\underline{s},\underline{s}) + e^{-\int_s^{\overline{s}} H^*(u)du} \left(\overline{p} - c(\underline{s},\underline{s})\right), \tag{14}$$

for $s \in [\underline{s}, \overline{s})$.

The term $\int_s^{\overline{s}} \frac{dc(v,v)}{dv} e^{-\int_s^v H^*(u)du} dv$ corresponds to the mark-up. It follows from (12) that the mark-up is proportional to how sensitive the competitor's bid is to its signal, i.e. $p'(s)$. Thus, it is understandable that the mark-up increases when the competitor's cost is more sensitive to its signal, i.e. $\frac{dc(v,v)}{dv}$ is large. Given that $H^*(s)$ is proportional to the ratio of the quantity and price effects, it also makes sense that a high $H^*(s)$ results in more competitive bids with lower mark-ups. We also note from Definition 2 that $H^*(s)$ and $p(s)$ are determined by the expected sales of the high price bidder and the low price bidder, but $H^*(s)$ and $p(s)$ are independent of the variances of those sales. The reason is that, by assumption, signals are independent of production capacities and demand. In the limit when firms' marginal costs are almost surely common knowledge, as in (14), the signals only serve the purpose of coordinating producers' actions as in a correlated equilibrium (Osborne and Rubinstein, 1994).

Another conclusion that we can draw from Proposition 1 is that bidding behaviour is only influenced by properties of $c_i(s_i, s_j)$ at points where $s_i = s_j$. Thus, it does not matter for our analysis whether the costs are private, so that $\frac{\partial c_i(s_i,s_j)}{\partial s_j} = 0$, or common, so that $\frac{\partial c_i(s_i,s_j)}{\partial s_j} = \frac{\partial c_i(s_i,s_j)}{\partial s_i}$. As noted above, the reason is that when solving for the locally optimal bid, a producer is only interested in cases where the competitor is bidding really close to p_i. In a symmetric equilibrium, this occurs when the competitor receives a similar signal. The properties of $c(\cdot)$ for signals where $s_i \neq s_j$ could influence the expected production cost of a firm, but not its bidding behaviour. This would be different if each bidder submitted multiple bids at different prices or even a continuous bid function as in Vives (2011).

Before drawing further conclusions from Proposition 1, we introduce the following definition:

Definition 3 *We say that two pairs of probability density functions and marginal cost functions* $\left\{\chi^A(s_i,s_j), c^A(s_i,s_j)\right\}$ *and* $\left\{\chi^B(s_i,s_j), c^B(s_i,s_j)\right\}$ *are equivalent in expectation if:*

(i) the pairs have the same expected marginal cost conditional on a producer's private signal s

$$\mathbb{E}\left[c^A(s,s_j)\middle|\, s\right] = \mathbb{E}\left[c^B(s,s_j)\middle|\, s\right],$$

(ii) the same marginal cost for common signals s

$$c^A(s,s) = c^B(s,s),$$

(iii) and the same marginal density

$$\int_{\underline{s}}^{\overline{s}} \chi^A(s,s_j)\, ds_j = \int_{\underline{s}}^{\overline{s}} \chi^B(s,s_j)\, ds_j.$$

It can be shown from Definition 2, (5) and (6) that $H^*(u)$ increases with respect to the production capacity $\widetilde{q}$. The reason is that the quantity effect increases when the difference between the output of the highest and the lowest bidder increases. It also follows from Definition 2 that $H^*(u)$ increases when the density at $\chi(s,s)$ increases relative to both $\int_{\underline{s}}^{s} \chi(s,s_j)\, ds_j$ and $\int_{s}^{\overline{s}} \chi(s,s_j)\, ds_j$. The reason is simply that the quantity effect from increasing one's bid increases if, conditional on the reception of a signal s, it becomes more likely that the competitor receives a similar signal s and bids at a similar price. Thus, we can conclude from Proposition 1 that

Corollary 1 *Mark-ups in the discriminatory auction are lower when $\widetilde{q}$ increases and are lower for the pair $\left\{\chi^A(s_i,s_j), c^A(s_i,s_j)\right\}$ in comparison to the pair $\left\{\chi^B(s_i,s_j), c^B(s_i,s_j)\right\}$, if the two pairs are equivalent in expectation and if signals in $\chi^A(s_i,s_j)$ are more positively correlated signals in the sense that*

$$\frac{\chi^A(s,s)}{\int_{\underline{s}}^{s} \chi^A(s,s_j)\, ds_j} > \frac{\chi^B(s,s)}{\int_{\underline{s}}^{s} \chi^B(s,s_j)\, ds_j}$$

and

$$\frac{\chi^A(s,s)}{\int_{s}^{\overline{s}} \chi^A(s,s_j)\, ds_j} > \frac{\chi^B(s,s)}{\int_{s}^{\overline{s}} \chi^B(s,s_j)\, ds_j}.$$

Proposition 1 can be simplified in the special case when signals are independent.

Proposition 2 *If signals are independent, the symmetric Bayesian Nash equilibrium bid in a discriminatory auction is given by:*

$$p(s) = c(s,s) + \int_{s}^{\overline{s}} \frac{dc(v,v)}{dv} \left(\frac{(1-F(v))\, q_H + F(v)\, q_L}{(1-F(s))\, q_H + F(s)\, q_L} \right) dv. \qquad (15)$$

If costs are almost surely common knowledge, then (15) can be simplified to:

$$p(s) = c(\underline{s},\underline{s}) + \left(\frac{q_L}{((1-F(s))\, q_H + F(s)\, q_L)} \right) (\overline{p} - c(\underline{s},\underline{s})) \text{ for } s \in [\underline{s},\overline{s}), \qquad (16)$$

where $F(s) = s$ is a firm's (marginal) probability distribution for receiving the signal s.

In the limit when costs are almost surely independent of the signals, the independent signals effectively become randomization devices, which the producers use to randomize their bids. In this case, the functional form of the probability density for signals is of no importance, because a firm will decide its bid based on the probability that the competitor received a lower signal, $F(s)$. Thus, bidding behaviour would not change if the probability distribution were transformed by the monotonic function $p(s)$ into a new signal $P = p(s)$ with the probability distribution $G(P) = F(p(s))$. If we rewrite (16), we get that

$$G(P) = \frac{q_H}{q_H - q_L} - \frac{\overline{p} - c}{P - c}\frac{q_L}{q_H - q_L}, \tag{17}$$

which corresponds to the mixed-strategy NE that is calculated for discriminatory auctions by Fabra et al. (2006). This confirms Harsanyi's (1973) purification theorem that a mixed-strategy NE is equivalent to a pure-strategy Bayesian NE, where costs are almost surely common knowledge and signals are independent.

Finally, we note that divisible-good models of discriminatory auctions, where each producer makes a single bid, are identical to the Bertrand model. Thus, the results in this section are also relevant for the Bertrand-Edgeworth game.

4.1.1 Non-pivotal case

Only the lowest bid is accepted when $\widetilde{q}_i > D$ for all $i \in \{1, 2\}$ and all outcomes, so that producers are non-pivotal with certainty, i.e. $q_L = 0$ and $q_H = \mathbb{E}[D]$. This simplifies the expressions to the below result, which corresponds to Milgrom and Weber's (1982) result for first-price indivisible-good auctions. If producers are non-pivotal with certainty, then the winning bid sets its own price also in the uniform-price auction. Thus, there is no difference between a discriminatory and a uniform-price auction in this special case. In this case, both correspond to the first-price auction in Milgrom and Weber (1982).

Proposition 3 *The symmetric Bayesian Nash equilibrium of producers that are non-pivotal with certainty in auctions with uniform or discriminatory pricing is given by:*

$$p(s) = c(s,s) + \int_s^{\overline{s}} \frac{dc(v,v)}{dv} e^{-\int_s^v H(u)du} dv, \tag{18}$$

where

$$H(s) := \frac{\chi(s,s)}{\int_s^{\overline{s}} \chi(s, s_j)\, ds_j}. \tag{19}$$

This is an equilibrium if the signals are weakly affiliated. The equilibrium exists for more general distributions when $\frac{dc(v,v)}{dv} > 0$. *In the limit when costs are almost surely common knowledge, the equilibrium bid in (18) is perfectly competitive, i.e.* $p(s) = c(\underline{s}, \underline{s})$ *for* $s \in [\underline{s}, \overline{s})$.

Mark-ups are zero in the limit when costs are almost surely common knowledge. This concurs with von der Fehr and Harbord (1993) and Fabra et al. (2006), where mark-ups are zero in auctions with both uniform and discriminatory pricing, if producers are non-pivotal with certainty and marginal costs are constant and common knowledge. Thus, their result for non-pivotal producers is robust to the probability distribution of weakly affiliated signals. The same robustness property applies to the Bertrand game. But private information gives an informational rent, so if costs are asymmetric information, then non-pivotal bidders also have a mark-up.

4.2 Uniform-pricing

Proposition 3 also applies to non-pivotal producers in a uniform-price auction. Below we consider producers that are pivotal with certainty. Later, we will consider the general case where producers are pivotal with a positive probability less than one.

Definition 4 *Producers are pivotal with certainty if it is always the case that* $\widetilde{q}_H < D \leq \widetilde{q}_H + \widetilde{q}_L$.

The highest bid sets the market price in a uniform-price auction when producers are pivotal with certainty. The demand and production capacity uncertainties are independent of the signals and cost uncertainties. Thus, when producers are pivotal with certainty, the expected profit of firm i when receiving signal s_i is:

$$\begin{aligned} \pi_i(s_i) = \mathbb{E}\left[p_j - c_i(s_i, s_j) \middle| p_j \geq p_i \right] \Pr\left(p_j \geq p_i \middle| s_i \right) q_H \\ + \left(p_i(s_i) - \mathbb{E}\left[c_i(s_i, s_j) \middle| p_j \leq p_i \right] \right) \left(1 - \Pr\left(p_j \geq p_i \middle| s_i \right) \right) q_L. \end{aligned} \tag{20}$$

Lemma 2 *In a uniform-price auction with producers that are pivotal with certainty, we have:*

$$\begin{aligned} \frac{\partial \pi_i(s_i)}{\partial p_i} = \left(1 - \Pr\left(p_j \geq p_i \middle| s_i \right) \right) q_L \\ + \frac{\partial \Pr(p_j \geq p_i | s_i)}{\partial p_i} \left(p_i - c_i\left(s_i, p_j^{-1}(p_i) \right) \right) (q_H - q_L). \end{aligned} \tag{21}$$

The first-order condition for the uniform-price auction is similar to the first-order condition of the discriminatory auction in Lemma 1, but there is one difference. In contrast to the discriminatory auction, the lowest bidder does not gain anything from increasing its bid in a uniform-price auction when producers are pivotal with certainty. Thus, the price effect has one term less in the uniform-price auction, which reduces the price effect. There is a corresponding change in the H function which is proportional to the ratio of the quantity and price effect. Thus, we introduce $\hat{H}(s)$.

Definition 5

$$\hat{H}(s) = \frac{(q_H - q_L)\, \chi(s, s)}{q_L \int_{\underline{s}}^{s} \chi(s, s_j)\, ds_j}.$$

Proposition 4 *The symmetric Bayesian Nash equilibrium bid in a uniform-price auction where producers are pivotal with certainty is given by*

$$p(s) = c(s,s) + \int_s^{\overline{s}} \frac{dc(v,v)}{dv} e^{-\int_s^v \hat{H}(u)du} dv \tag{22}$$

if signals are weakly unaffiliated. The equilibrium exists for more general probability distributions when $\frac{dc(v,v)}{dv} > 0$*. In the limit when costs are almost surely common knowledge, the symmetric Bayesian Nash equilibrium bid simplifies to:*

$$p(s) = c(\underline{s},\underline{s}) + e^{-\int_s^{\overline{s}} \hat{H}(u)du} (\overline{p} - c(\underline{s},\underline{s})), \tag{23}$$

for $s \in [\underline{s}, \overline{s})$*.*

(22) has properties similar to the corresponding expressions for the discriminatory auction in Proposition 1. But the ratio of the quantity and price effects differs. It follows from Definitions 2 and 5 that $\hat{H}(s) > H^*(s)$ or, equivalently, that the price effect is relatively smaller in the uniform price auction as compared to a discriminatory auction. Thus, producers bid with lower mark-ups in uniform-price auctions. On the other hand, in a uniform-price auction, the losing bid (the highest bid) sets the transaction price for both accepted bids, so in the end it is not self-evident that a uniform-price auction would lower the procurement cost of an auctioneer.

Analogous to the discriminatory case, it can be shown from Definition 5, (5) and (6) that $\hat{H}(u)$ increases with respect to the production capacity $\widetilde{q}$. It also follows from Definition 5 that $\hat{H}(u)$ increases when the density at $\chi(s,s)$ increases relative to $\int_{\underline{s}}^s \chi(s,s_j)\,ds_j$. Thus, we can conclude from Proposition 4 and Definition 3 that

Corollary 2 *Mark-ups in a uniform-price auction where producers are pivotal with certainty are lower when* $\widetilde{q}$ *increases and are lower for the pair* $\{\chi^A(s_i,s_j), c^A(s_i,s_j)\}$ *in comparison to the pair* $\{\chi^B(s_i,s_j), c^B(s_i,s_j)\}$*, if the two pairs are equivalent in expectation and if the signals in* $\chi^A(s_i,s_j)$ *are more positively correlated in the sense that*

$$\frac{\chi^A(s,s)}{\int_{\underline{s}}^s \chi^A(s,s_j)\,ds_j} > \frac{\chi^B(s,s)}{\int_{\underline{s}}^s \chi^B(s,s_j)\,ds_j}.$$

It follows from Definition 5, (5), (6) and Proposition 4 that

Corollary 3 *In a uniform-price auction with certain demand and production capacities, it is optimal for firm i to bid:*
i) $p_i(s) = c_i(s_i,s_i)$ *in the limit where* $\widetilde{q} \nearrow D$*, i.e. when firms are just pivotal.*
ii) $p_i(s) = \overline{p}$ *in the limit where* $2\widetilde{q} \searrow D$*, i.e. when both firms always produce at full capacity.*

The first property corresponds to Milgrom and Weber's (1982) results for second-price sales auctions, because the lowest bidder gets to produce the whole demand while the highest bidder sets the uniform market price. By comparing Proposition 3 and Corollary 3, we note that the comparative statics analysis of our symmetric equilibrium has a discontinuity at the critical point where producers' capacities switch from being nonpivotal with certainty to being pivotal with certainty. Somewhat counter-intuitively, bids decrease at this critical point, even if demand increases. The reason for this is that the bid that sets the market price also switches at this point, which drastically changes the bidding behaviour. Non-pivotal firms set their own price and use similar bidding strategies as in a first-price procurement auction, i.e. firms' mark-ups are strictly positive for uncertain costs. On the other hand, as implied by the first property of Corollary 3, producers bid without a mark-up when firms are just pivotal. The following proves that the auctioneer's revenues may also shift downwards in a comparative statics analysis at the critical point where producers' capacities switch from being nonpivotal with certainty to being pivotal with certainty.

Proposition 5 *If firms' signals are weakly affiliated, then the expected revenues of the auctioneer are weakly larger for just non-pivotal producers than for producers that are just pivotal with certainty in markets with uniform pricing. The expected revenues are the same for the two cases when the signals are independent .*

Proposition 4 can be simplified in the special case when the signals are independent.

Proposition 6 *The symmetric Bayesian Nash equilibrium bid in a uniform-price auction where producers are pivotal with certainty is given by:*

$$p(s) = c(s,s) + \int_s^{\overline{s}} \frac{dc(v,v)}{dv} \left(\frac{F(s)}{F(v)} \right)^{\frac{(q_H - q_L)}{q_L}} dv \tag{24}$$

if signals are independent. If, in addition, costs are almost surely common knowledge, then (24) can be simplified to

$$p(s) = c(\underline{s},\underline{s}) + (F(s))^{\frac{(q_H - q_L)}{q_L}} (\overline{p} - c(\underline{s},\underline{s})) \text{ for } s \in [\underline{s},\overline{s}), \tag{25}$$

where $F(s)$ is a firm's marginal distribution for receiving the signal s.

We can use an argument similar to the one that we used for the discriminatory auction to show that the limit result in (25) corresponds to the mixed-strategy NE that is derived for uniform-price auctions by von der Fehr and Harbord (1993).

Proposition 7 *If the signals are independent, the costs are almost surely common knowledge and producers are pivotal with certainty, then the expected market price in the uniform-price auction is given by:*

$$\overline{p} - \frac{(\overline{p} - c)(q_H - q_L)}{q_H + q_L},$$

where $c = c(\underline{s},\underline{s})$.

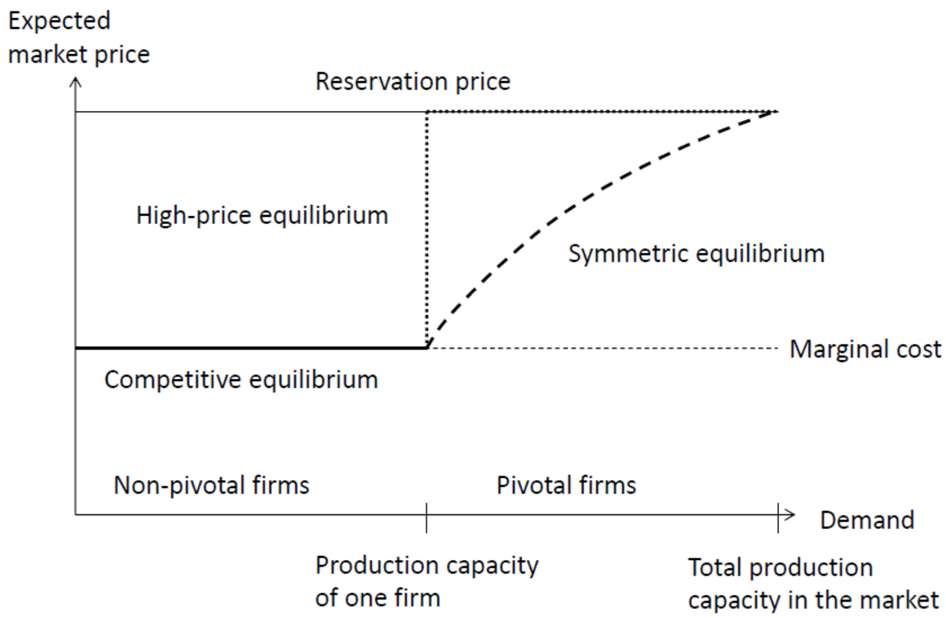

Figure 1: Comparative statics analysis for our symmetric equilibrium and von der Fehr and Harbord's (1993) high-price equilibrium in a uniform-price auction.

In the special case with certain demand and certain production capacities that are pivotal, we have $q_H = \widetilde{q}$ and $q_L = D - \widetilde{q} > 0$, so that the expected market price is given by

$$\overline{p} - \frac{(\overline{p} - c)(2\widetilde{q} - D)}{D}. \tag{26}$$

Our symmetric Bayesian NE for the uniform-price auction has similar properties to von der Fehr and Harbord's (1993) symmetric mixed-strategy NE, especially when the cost uncertainty is small and the signals are independent in our setting. However, their asymmetric high-price equilibrium has quite different properties. Figure 1 presents comparative statics analyses of the symmetric and high-price equilibria with respect to the (expected) demand level when the costs are almost surely common knowledge. In the high-price equilibrium, the market price jumps directly from the competitive price with zero mark-ups up to the reservation price when demand increases at the critical point where producers switch from being non-pivotal to being pivotal. In the symmetric equilibrium, on the other hand, the expected market price increases continuously as demand increases. The expected market price does not reach the reservation price until demand equals the total production capacity in the market. With more firms in the market, the expected price in our model would stay near the marginal cost until demand is near the total production capacity in the market, where the expected price will take off towards the reservation price. This would be reminiscent of hockey-stick pricing that is typical for wholesale electricity markets (Hurlbut et al., 2004; Holmberg and Newbery, 2010).

4.2.1 Uncertain pivotal status

There is no difference in the discriminatory case, but for a uniform-price auction the case where a producer is both pivotal and non-pivotal with a strictly positive probability is more complicated than the case where producers are pivotal with certainty. The problem is that the lowest bidder would set its own transaction price, as in a discriminatory auction, for outcomes when the highest bidder is non-pivotal, while the highest bidder would set the transaction price of the lowest bidder when the highest bidder is pivotal. Thus, unlike the discriminatory auction, the payoff of the winning producer depends on the probability that the highest bidder is non-pivotal. We denote this probability by Π^{NP}.

Lemma 3 *In a uniform-price auction, where producers can be pivotal and non-pivotal with positive probabilities, we have:*

$$\begin{array}{c}\frac{\partial \pi_i(s_i)}{\partial p_i} = \frac{\partial \Pr(p_j \geq p_i | s_i)}{\partial p_i} \left(p_i - c_i\left(s_i, p_j^{-1}(p_i)\right)\right)(q_H - q_L) \\ + \Pr(p_j \geq p_i | s_i)\, q_H^{NP} \Pi^{NP} + (1 - \Pr(p_j \geq p_i | s_i))\, q_L,\end{array} \tag{27}$$

where

$$q_H^{NP} = \mathbb{E}\left[\widetilde{q}_H \middle| \widetilde{q}_H \geq D\right].$$

Thus, the quantity effect is the same as when producers are pivotal or non-pivotal with certainty. But the price effect depends on the probability that firms are pivotal. Increasing a bid contributes to the price effect when the bidder is price-setting, i.e. when it has the highest bid and the highest bidder is pivotal ($q_L > 0$) or when it has the lowest bid and the highest bidder is non-pivotal. There is a corresponding change in the H function.

Definition 6

$$\widetilde{H}(s) = \frac{\chi(s,s)(q_H - q_L)}{\int_s^{\overline{s}} \chi(s, s_j)\, ds_j q_H^{NP} \Pi^{NP} + \int_{\underline{s}}^{s} \chi(s, s_j)\, ds_j q_L}.$$

Proposition 8 *The symmetric Bayesian Nash equilibrium bid in a uniform-price auction where producers can be pivotal and non-pivotal with positive probabilities is given by*

$$p(s) = c(s,s) + \int_s^{\overline{s}} \frac{dc(v,v)}{dv} e^{-\int_s^v \widetilde{H}(u)du} dv \tag{28}$$

if $\frac{d}{ds}\left(\frac{\int_x^{\overline{s}} \chi(s,s_j)ds_j q_H^{NP}\Pi^{NP} + q_L \int_{\underline{s}}^x \chi(s,s_j)ds_j}{\chi(s,x)}\right) \geq 0$. *The equilibrium exists for more general probability distributions when* $\frac{dc(v,v)}{dv} > 0$. *In the limit when costs are almost surely common knowledge, the symmetric Bayesian Nash equilibrium bid simplifies to:*

$$p(s) = c(\underline{s}, \underline{s}) + e^{-\int_s^{\overline{s}} \widetilde{H}(u)du} (\overline{p} - c(\underline{s}, \underline{s})), \tag{29}$$

for $s \in [\underline{s}, \overline{s})$.

We note that as Π^{NP} increases towards 1, the bidding behaviour in the uniform-price auction gets closer to bids in the discriminatory auction, which concurs with our discussion in Section 4.1.1. In the other extreme, bidding gets closer to the uniform-price auction with producers that are pivotal with certainty when Π^{NP} decreases towards 0. For a given q_H^{NP}, producers will increase their bids when Π^{NP} increases. This may seem counterintuitive, but this is to compensate for the fact that there is a higher risk that the market price is set by the lowest bid rather than the highest bid. We can draw the following conclusion from Proposition 8 and Definition 6.

Corollary 4 *Mark-ups in an auction with uniform-pricing are lower when $\widetilde{q}$ increases and are lower for the pair $\left\{\chi^A(s_i,s_j), c^A(s_i,s_j)\right\}$ in comparison to the pair $\left\{\chi^B(s_i,s_j), c^B(s_i,s_j)\right\}$, if the two pairs are equivalent in expectation and if the signals in $\chi^A(s_i,s_j)$ are more positively correlated signals in the sense that*

$$\frac{\chi^A(s,s)}{\int_{\underline{s}}^{s}\chi^A(s,s_j)\,ds_j} > \frac{\chi^B(s,s)}{\int_{\underline{s}}^{s}\chi^B(s,s_j)\,ds_j}$$

and

$$\frac{\chi^A(s,s)}{\int_{s}^{\overline{s}}\chi^A(s,s_j)\,ds_j} > \frac{\chi^B(s,s)}{\int_{s}^{\overline{s}}\chi^B(s,s_j)\,ds_j}.$$

Proposition 4 can be simplified in the special case when signals are independent.

Proposition 9 *The symmetric Bayesian Nash equilibrium bid in a uniform-price auction where producers can be pivotal or nonpivotal with positive probabilities is given by:*

$$p(s) = c(s,s) + \int_{s}^{\overline{s}} \frac{dc(v,v)}{dv}\left(\frac{(1-F(v))\,q_H^{NP}\Pi^{NP} + F(v)\,q_L}{(1-F(s))\,q_H^{NP}\Pi^{NP} + F(s)\,q_L}\right)^{\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}} dv \tag{30}$$

if the signals are independent. If, in addition, costs are almost surely common knowledge, then (24) can be simplified to

$$p(s) = c(\underline{s},\underline{s}) + \left(\frac{q_L}{((1-F(s))\,q_H^{NP}\Pi^{NP} + F(s)\,q_L)}\right)^{\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}} (\overline{p} - c(\underline{s},\underline{s})) \quad \text{for } s \in [\underline{s},\overline{s}) \tag{31}$$

where $F(s)$ is a firm's marginal distribution for receiving the signal s.

4.3 Ranking of auction formats

We already know from Section 4.1.1 that the two auction formats are equivalent in the non-pivotal case. The following result shows that there are cases where the two auction formats are equivalent also when producers are pivotal with a positive probability, so that $q_L > 0$.

Proposition 10 *If signals are independent and costs are almost surely common knowledge, then the expected profit for a producer is given by*

$$\pi(s) = q_L\left(\overline{p} - c\left(\underline{s}, \underline{s}\right)\right),$$

for both the uniform-price and the discriminatory auction and irrespective of the probability that the highest bidder is pivotal.

However, an auctioneer tends to prefer the uniform-price auction if the costs are uncertain.

Proposition 11 *If signals are independent and costs are uncertain, then an auctioneer will (weakly) prefer a uniform-price auction. For fixed expected sales of the winning and losing bidder, a higher probability that the winning bidder is pivotal strictly lowers the auctioneer's payoff in a uniform-price auction, but does not influence the payoffs in a discriminatory auction.*

For a given marginal distribution $F(s)$, we can from Definition 2 and Proposition 1 identify circumstances where a joint probability distribution $\chi(u, v)$ will give a higher bid in the discriminatory auction in comparison to the case with independent signals, where $\chi(u, v) = f(u) f(v)$. Similarly, for the same marginal distribution $F(s)$, we can use Definition 6 and Proposition 8 to identify circumstances where a joint probability distribution $\chi(u, v)$ will give a lower bid in the uniform-price auction in comparison to the case with independent signals, where $\chi(u, v) = f(u) f(v)$. Together with Proposition 11, such an argument gives the following result:

Corollary 5 *Provided that the cost uncertainty is sufficiently large so that an equilibrium exists in both auctions, an auctioneer will (weakly) prefer a uniform-price auction to a discriminatory auction if the joint probability density of signals $\chi(u, v)$ with the marginal distribution $F(u)$ satisfy:*

$$\frac{(1 - F(s))\, q_H^{NP}\Pi^{NP} + F(s)\, q_L}{\int_s^{\overline{s}} \chi(s, s_j)\, ds_j q_H^{NP}\Pi^{NP} + \int_{\underline{s}}^{s} \chi(s, s_j)\, ds_j q_L} \leq \frac{f(s)}{\chi(s, s)} \leq \frac{(1 - F(s))\, q_H + F(s)\, q_L}{\int_s^{\overline{s}} \chi(s, s_j)\, ds_j q_H + \int_{\underline{s}}^{s} \chi(s, s_j)\, ds_j q_L}. \tag{32}$$

for all s.

We have $q_H \geq q_H^{NP}\Pi^{NP}$, so we note that a necessary condition for (32) to be satisfied is that $\frac{(1-F(s))}{f(s)} \geq \frac{\int_s^{\overline{s}} \chi(s,s_j) ds_j}{\chi(s,s)}$ and that $\frac{F(s)}{f(s)} \leq \frac{\int_{\underline{s}}^{s} \chi(s,s_j) ds_j}{\chi(s,s)}$ for all s. The condition is satisfied for a larger set of joint probability densities $\chi(u, v)$ when the probability that the winning bidder is pivotal increases, so that $q_H^{NP}\Pi^{NP}$ decreases, for a given q_H.

5 Concluding discussion

We consider a duopoly model of a divisible-good procurement auction with production uncertainty, such as a wholesale electricity market. Each producer receives a private signal with imperfect cost information from a non-parametric probability distribution and then makes one bid for its whole production capacity. Demand and production capacities could also be uncertain. However, producers do not have any private information about such events.

We solve for a symmetric Bayesian NE. We show that the auctioneer prefers uniform pricing to discriminatory pricing when producers' signals are close to independent. The advantages of uniform-price auctions increase when cost uncertainty increases and when firms are pivotal with a higher probability. Moreover, competition improves if producers receive less noisy cost information. The latter concurs with a similar result for another bidding format by Vives (2011), and Milgrom and Weber's (1982) result for single object auctions. Taken together, this supports the measures taken by the European Commission to increase transparency in European wholesale electricity markets.

We are concerned that cost uncertainty could result in major mark-ups in hydro dominated electricity markets with scarce water. This could explain extraordinarily high price-periods that typically accompany scarcity of water in such markets. One measure that could mitigate this is to clearly define contingency plans for intervention by the market operator and the regulator under extreme system conditions.

If producers are pivotal, then disclosure of information is only beneficial up to a point. A pivotal producer can deviate to the reservation price, which ensures it a minimum profit. Thus, there is a lower bound on how small equilibrium mark-ups can become. Mark-ups tend to be higher in the discriminatory auction in comparison to the uniform-price auction, so disclosure of information should be somewhat more useful in such auctions. Our results and related results in Vives (2011) and Milgrom and Weber (1982) have been proven for a one-shot game. In a repeated game, there is a risk that increased transparency will facilitate tacit collusion as argued by von der Fehr (2013).

We show that equilibrium bids in a discriminatory auction are determined by the expected sales of the highest and lowest bidder, respectively. The variance of these sales – due to demand shocks, production outages and volatile renewable production – will not influence the bidding behaviour of producers. Bidding in the uniform-price auction is also insensitive to these variances, as long as they are not sufficiently large to occasionally change the pivotal status of at least one producer.

Unlike Vives (2011), our results do not depend on the extent to which the cost uncertainty is private, interdependent or common. In his setting, producers choose linear supply functions and can therefore condition their output on every price. To a larger extent than in our model, his bidding format allows producers to condition their output on the competitor's information. Thus, our results and the results in Vives (2011) indicate that when the cost uncertainty is common or strongly interdependent, which should often be the case in wholesale electricity markets, then it should be optimal to limit the number of allowed steps in producers' supply functions in order to give producers less freedom to condition their output on competitors' signals. This also suggests that bidding formats, where producers make piece-wise linear bids, as in the Nordic countries (Nord Pool) and France (Power Next), can be harmful for market competitiveness.

We consider unique symmetric equilibria. Our general analysis is for cases where producers are not certain whether they are pivotal or not. But as shown by von der Fehr and Harbord (1993), the uniform-price auction has another asymmetric, high-price equilibrium if producers are certain that they are pivotal. Fabra et al. (2006) consider the high-price equilibrium in the uniform-price auction when comparing auction formats. This explains why they get a different ranking of the uniform and discriminatory price auctions than we do.

Similar to von der Fehr and Harbord (1993), our results also indicate that wholesale electricity markets that require producers to submit stepped supply function bids have an inherent price instability. Even if the cost uncertainty in our model were arbitrarily small, the chosen bid price of a producer can still increase drastically if it receives a signal that indicates that the producer's costs are slightly higher. Thus, according to our model, observed price volatilities would be larger than what would be expected from variations in demand and costs. This is true for both the uniform-price and the discriminatory auction. However, as conjectured by Newbery (1998), it seems plausible that the inherent price instability of the market design will become smaller if producers, as is the case in practice, are allowed to choose a larger number of steps in their supply functions.

6 References

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Appendix

Before proving the lemmas and propositions in the main text, we will derive some results that will be used throughout these proofs. By assumption, $p_j(s_j)$

is monotonic and invertible. Thus, we get

$$\begin{aligned} \Pr\left(p_j \geq p_i \middle| s_i\right) &= \frac{\int_{p_j^{-1}(p_i)}^{\overline{s}} \chi(s_i,s_j)ds_j}{\int_{\underline{s}}^{\overline{s}} \chi(s_i,s_j)ds_j} \\ \frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} &= \frac{-p_j^{-1\prime}(p_i)\chi\left(s_i,p_j^{-1}(p_i)\right)}{\int_{\underline{s}}^{\overline{s}} \chi(s_i,s_j)ds_j}. \end{aligned} \tag{33}$$

Moreover,

$$\begin{aligned} \mathbb{E}\left[c_i\left(s_i,s_j\right)\middle| p_j \geq p_i\right] &= \frac{\int_{p_j^{-1}(p_i)}^{\overline{s}} c_i(s_i,s_j)\chi(s_i,s_j)ds_j}{\int_{p_j^{-1}(p_i)}^{\overline{s}} \chi(s_i,s_j)ds_j} = \frac{\int_{p_j^{-1}(p_i)}^{\overline{s}} c_i(s_i,s_j)\chi(s_i,s_j)ds_j}{\Pr(p_j \geq p_i|s_i)\int_{\underline{s}}^{\overline{s}} \chi(s_i,s_j)ds_j} \\ \frac{\partial \mathbb{E}[c_i(s_i,s_j)|p_j \geq p_i]}{\partial p_i} &= \frac{p_j^{-1\prime}(p_i)\chi\left(s_i,p_j^{-1}(p_i)\right)\int_{p_j^{-1}(p_i)}^{\overline{s}}\left(c_i(s_i,s_j)-c_i\left(s_i,p_j^{-1}(p_i)\right)\right)\chi(s_i,s_j)ds_j}{\left(\int_{p_j^{-1}(p_i)}^{\overline{s}} \chi(s_i,s_j)ds_j\right)^2} \\ &= \frac{-\frac{\partial \Pr\left(p_j \geq p_i\middle|s_i\right)}{\partial p_i}\int_{p_j^{-1}(p_i)}^{\overline{s}}\left(c_i(s_i,s_j)-c_i\left(s_i,p_j^{-1}(p_i)\right)\right)\chi(s_i,s_j)ds_j}{\left(\Pr(p_j \geq p_i|s_i)\right)^2\int_{\underline{s}}^{\overline{s}} \chi(s_i,s_j)ds_j}. \end{aligned} \tag{34}$$

From (33) and (34), we have that:

$$\begin{aligned} &-\frac{\partial \mathbb{E}[c_i(s_i,s_j)|p_j \geq p_i]}{\partial p_i}\Pr\left(p_j \geq p_i\middle| s_i\right) - \mathbb{E}\left[c_i\left(s_i,s_j\right)\middle| p_j \geq p_i\right]\frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} \\ &= \left(\frac{\int_{p_j^{-1}(p_i)}^{\overline{s}}\left(c_i(s_i,s_j)-c_i\left(s_i,p_j^{-1}(p_i)\right)\right)\chi(s_i,s_j)ds_j}{\int_{p_j^{-1}(p_i)}^{\overline{s}} \chi(s_i,s_j)ds_j} - \frac{\int_{p_j^{-1}(p_i)}^{\overline{s}} c_i(s_i,s_j)\chi(s_i,s_j)ds_j}{\int_{p_j^{-1}(p_i)}^{\overline{s}} \chi(s_i,s_j)ds_j}\right)\frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} \\ &= -\frac{\int_{p_j^{-1}(p_i)}^{\overline{s}} c_i\left(s_i,p_j^{-1}(p_i)\right)\chi(s_i,s_j)ds_j}{\int_{p_j^{-1}(p_i)}^{\overline{s}} \chi(s_i,s_j)ds_j}\frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} \\ &= -c_i\left(s_i,p_j^{-1}\left(p_i\right)\right)\frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i}. \end{aligned}$$

Using the above equation, we can derive the following result:

$$\begin{aligned} &\left(1-\frac{\partial \mathbb{E}[c_i(s_i,s_j)|p_j \geq p_i]}{\partial p_i}\right)\Pr\left(p_j \geq p_i\middle| s_i\right) + \left(p_i - \mathbb{E}\left[c_i\left(s_i,s_j\right)\middle| p_j \geq p_i\right]\right)\frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} \\ &= \Pr\left(p_j \geq p_i\middle| s_i\right) + \left(p_i - c_i\left(s_i,p_j^{-1}\left(p_i\right)\right)\right)\frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i}. \end{aligned} \tag{35}$$

Similarly, from (33), we have that

$$\begin{aligned} \mathbb{E}\left[c_i\left(s_i,s_j\right)\middle| p_j \leq p_i\right] &= \frac{\int_{\underline{s}}^{p_j^{-1}(p_i)} c_i(s_i,s_j)\chi(s_i,s_j)ds_j}{\int_{\underline{s}}^{p_j^{-1}(p_i)} \chi(s_i,s_j)ds_j} = \frac{\int_{\underline{s}}^{p_j^{-1}(p_i)} c_i(s_i,s_j)\chi(s_i,s_j)ds_j}{(1-\Pr(p_j \geq p_i|s_i))\int_{\underline{s}}^{\overline{s}} \chi(s_i,s_j)ds_j} \\ \frac{\partial \mathbb{E}[c_i(s_i,s_j)|p_j \leq p_i]}{\partial p_i} &= \frac{p_j^{-1\prime}(p_i)\chi\left(s_i,p_j^{-1}(p_i)\right)\int_{\underline{s}}^{p_j^{-1}(p_i)}\left(c_i\left(s_i,p_j^{-1}(p_i)\right)-c_i(s_i,s_j)\right)\chi(s_i,s_j)ds_j}{\left(\int_{\underline{s}}^{p_j^{-1}(p_i)} \chi(s_i,s_j)ds_j\right)^2} \\ &= \frac{-\frac{\partial \Pr\left(p_j \geq p_i\middle|s_i\right)}{\partial p_i}\int_{\underline{s}}^{p_j^{-1}(p_i)}\left(c_i\left(s_i,p_j^{-1}(p_i)\right)-c_i(s_i,s_j)\right)\chi(s_i,s_j)ds_j}{(1-\Pr(p_j \geq p_i|s_i))^2\int_{\underline{s}}^{\overline{s}} \chi(s_i,s_j)ds_j}. \end{aligned} \tag{36}$$

It now follows from (36) that:

$$\begin{aligned} &-\frac{\partial \mathbb{E}[c_i(s_i,s_j)|p_j \leq p_i]}{\partial p_i}\left(1-\Pr\left(p_j \geq p_i\middle| s_i\right)\right) + \mathbb{E}\left[c_i\left(s_i,s_j\right)\middle| p_j \leq p_i\right]\frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} \\ &= \frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i}c_i\left(s_i,p_j^{-1}\left(p_i\right)\right). \end{aligned} \tag{37}$$

Discriminatory auction

Proof. (Lemma 1) It follows from (7) that

$$\begin{aligned}\frac{\partial \pi_i(s_i)}{\partial p_i} = &\left(1 - \frac{\partial \mathbb{E}[c_i(s_i,s_j)|p_j \geq p_i]}{\partial p_i}\right) \Pr\left(p_j \geq p_i|\, s_i\right) q_H \\ &+ \left(p_i - \mathbb{E}\left[c_i\left(s_i, s_j\right)|\, p_j \geq p_i\right]\right) \frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} q_H \\ &+ \left(1 - \frac{\partial \mathbb{E}[c_i(s_i,s_j)|p_j \leq p_i]}{\partial p_i}\right) \left(1 - \Pr\left(p_j \geq p_i|\, s_i\right)\right) q_L \\ &- \left(p_i - \mathbb{E}\left[c_i\left(s_i, s_j\right)|\, p_j \leq p_i\right]\right) \frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} q_L.\end{aligned} \tag{38}$$

Using (35) and the relation in (37) yields:

$$\begin{aligned}\frac{\partial \pi_i(s_i)}{\partial p_i} = &\Pr\left(p_j \geq p_i|\, s_i\right) q_H + \left(p_i - c_i\left(s_i, p_j^{-1}\left(p_i\right)\right)\right) \frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} q_H \\ &+ c_i\left(s_i, p_j^{-1}\left(p_i\right)\right) \frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} q_L \\ &+ \left(1 - \Pr\left(p_j \geq p_i|\, s_i\right)\right) q_L - p_i \frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} q_L,\end{aligned}$$

which gives (8). ■

Proof. (Proposition 1) We solve for symmetric strategies, so that $p_i(s) = p_j(s) = p(s)$. We also note that $p^{-1\prime}(p) = \frac{1}{p'(s)}$. We get the first-order condition from (8) and by setting $\frac{\partial \pi_i(s_i)}{\partial p_i} = 0$. Using (33), this condition can be written as follows:

$$\int_s^{\overline{s}} \chi\left(s, s_j\right) ds_j q_H + \int_{\underline{s}}^{s} \chi\left(s, s_j\right) ds_j q_L - \frac{\left(p - c\left(s, s\right)\right)}{p'\left(s\right)} \chi\left(s, s\right)\left(q_H - q_L\right) = 0.$$

We can use the definition in (11) to write the first-order condition on the following form:

$$p'\left(s\right) - \left(p - c\left(s, s\right)\right) H^*\left(s\right) = 0. \tag{39}$$

Multiplication by the integrating factor $e^{\int_s^{\overline{s}} H^*(u)du}$ yields:

$$\begin{aligned} & p'\left(s\right) e^{\int_s^{\overline{s}} H^*(u)du} - pH^*\left(s\right) e^{\int_s^{\overline{s}} H^*(u)du} \\ = & -c\left(s, s\right) H^*\left(s\right) e^{\int_s^{\overline{s}} H^*(u)du},\end{aligned}$$

so that

$$\frac{d}{ds}\left(p\left(s\right) e^{\int_s^{\overline{s}} H^*(u)du}\right) = -c\left(s, s\right) H^*\left(s\right) e^{\int_s^{\overline{s}} H^*(u)du}.$$

Next we integrate both sides from s to $\overline{s}$.

$$\begin{aligned}\overline{p} - p\left(s\right) e^{\int_s^{\overline{s}} H^*(u)du} &= -\int_s^{\overline{s}} c\left(v, v\right) H^*\left(v\right) e^{\int_v^{\overline{s}} H^*(u)du}\, dv \\ p\left(s\right) &= \overline{p} e^{-\int_s^{\overline{s}} H^*(u)du} + \int_s^{\overline{s}} c\left(v, v\right) H^*\left(v\right) e^{-\int_s^{v} H^*(u)du} dv.\end{aligned}$$

We use integration by parts to rewrite the above expression as follows:

$$p\left(s\right) = \overline{p} e^{-\int_s^{\overline{s}} H^*(u)du} + \left[-c\left(v, v\right) e^{-\int_s^{v} H^*(u)du}\right]_s^{\overline{s}} + \int_s^{\overline{s}} \frac{dc\left(v, v\right)}{dv} e^{-\int_s^{v} H^*(u)du} dv,$$

which gives (13), because $c(\overline{s},\overline{s}) = \overline{p}$. It is clear from (13) that $p > c(s,s)$ for $s \in [\underline{s},\overline{s})$. Hence, it follows from (39) that $p'(s) > 0$ for $s \in [\underline{s},\overline{s})$.

It remains to show that $p(s)$ is an equilibrium. It follows from (8) and (33) that

$$\begin{aligned}
\frac{\partial \pi_i(s)}{\partial p} &= \frac{\int_{p_j^{-1}(p)}^{\overline{s}} \chi(s,s_j)\,ds_j}{\int_{\underline{s}}^{\overline{s}} \chi(s,s_j)\,ds_j} q_H + \frac{\int_{\underline{s}}^{p_j^{-1}(p)} \chi(s,s_j)\,ds_j}{\int_{\underline{s}}^{\overline{s}} \chi(s,s_j)\,ds_j} q_L \\
&\quad - \frac{p_j^{-1\prime}(p)\,\chi\left(s,p_j^{-1}(p)\right)}{\int_{\underline{s}}^{\overline{s}} \chi(s,s_j)\,ds_j}\left(p - c_i\left(s,p_j^{-1}(p)\right)\right)(q_H - q_L). \\
\frac{\partial \pi_i(s)}{\partial p} &= \frac{\chi\left(s,p_j^{-1}(p)\right)}{\int_{\underline{s}}^{\overline{s}} \chi(s,s_j)\,ds_j}\left(\frac{\int_{p_j^{-1}(p)}^{\overline{s}} \chi(s,s_j)\,ds_j}{\chi\left(s,p_j^{-1}(p)\right)} q_H + \frac{\int_{\underline{s}}^{p_j^{-1}(p_i)} \chi(s,s_j)\,ds_j}{\chi\left(s,p_j^{-1}(p)\right)} q_L\right. \\
&\quad \left. - p_j^{-1\prime}(p)\left(p - c_i\left(s,p_j^{-1}(p)\right)\right)(q_H - q_L)\right).
\end{aligned}$$

We know that $\frac{\partial \pi_i(s)}{\partial p} = 0$ for $s = p_j^{-1}(p)$. We assume that $\frac{d}{ds}\left(\frac{\int_x^{\overline{s}} \chi(s,s_j)ds_j q_H + q_L \int_{\underline{s}}^x \chi(s,s_j)ds_j}{\chi(s,x)}\right) \geq 0$. Thus, it follows from the above and (3) that $\frac{\partial \pi_i(s)}{\partial p} > 0$ when $s > p_j^{-1}(p) \iff p < p_j(s)$ and that $\frac{\partial \pi_i(s)}{\partial p} < 0$ when $s < p_j^{-1}(p) \iff p > p_j(s)$. Thus, $p(s)$ maximizes the profit of firm i.

In the special case when costs are almost surely common knowledge, we have $\frac{dc(v,v)}{dv} \searrow 0$ for $v < \overline{s}$, so it follows from (13) that

$$p(s) \to c(\underline{s},\underline{s}) + e^{-\int_s^{\overline{s}} H^*(u)du} \int_s^{\overline{s}} \frac{dc(v,v)}{dv} dv,$$

which gives (14).

(Proposition 2) First we note that $\chi(s,s_j) = f(s) f(s_j)$ for independent signals, so the inequality

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{\int_x^{\overline{s}} \chi(s,s_j)\,ds_j q_H + q_L \int_{\underline{s}}^x \chi(s,s_j)\,ds_j}{\chi(s,x)}\right) \\
= &\frac{d}{ds}\left(\frac{\int_x^{\overline{s}} f(s) f(s_j)\,ds_j q_H + q_L \int_{\underline{s}}^x f(s) f(s_j)\,ds_j}{f(s) f(x)}\right) = \\
= &\frac{d}{ds}\left(\frac{\int_x^{\overline{s}} f(s_j)\,ds_j q_H + q_L \int_{\underline{s}}^x f(s_j)\,ds_j}{f(x)}\right) = 0 \geq 0
\end{aligned}$$

is satisfied. Thus, it follows from Proposition 1 that the global second-order condition is satisfied. Moreover, for independent signals, we have from Definition 2 that

$$\begin{aligned}
H^*(s) &= \frac{f(s)(q_H - q_L)}{\int_s^{\overline{s}} f(s_j)\,ds_j q_H + \int_{\underline{s}}^s f(s_j)\,ds_j q_L} \\
&= -\frac{d}{ds}\ln\left(\int_s^{\overline{s}} f(s_j)\,ds_j q_H + \int_{\underline{s}}^s f(s_j)\,ds_j q_L\right).
\end{aligned}$$

Thus, (13) can be written as in (15).

In case costs are almost surely common knowledge, so that $\frac{dc(v,v)}{dv} \searrow 0$ for $v < \overline{s}$, (15) can be simplified to (16) as follows:

$$\begin{aligned} p(s) &= c(\underline{s},\underline{s}) + \left(\frac{\int_{\underline{s}}^{\overline{s}} \chi f(s_j)\, ds_j q_L}{\int_s^{\overline{s}} f(s_j)\, ds_j q_H + \int_{\underline{s}}^{s} f(s_j)\, ds_j q_L} \right) \int_s^{\overline{s}} \frac{dc(v,v)}{dv} dv \\ &= c(\underline{s},\underline{s}) + \left(\frac{q_L}{((1-F(s))\, q_H + F(s)\, q_L)} \right) (\overline{p} - c(\underline{s},\underline{s})), \end{aligned}$$

where $F(s)$ is the marginal probability distribution. ■

Non-pivotal case

The following lemma is useful when deriving results for the non-pivotal case.

Lemma 4 *$e^{-\int_s^v H(u)du} > 0$ for $\underline{s} \leq s < v < \overline{s}$ and $e^{-\int_s^v H(u)du} = 0$ for $\underline{s} \leq s < v = \overline{s}$.*

Proof. It follows from (19) that

$$H(u) = \frac{\chi(u,u)}{\int_u^{\overline{s}} \chi(u,s_j)\, ds_j} = -\frac{d}{du} \ln \left(\int_u^{\overline{s}} \chi(u,s_j)\, ds_j \right) + \frac{\int_u^{\overline{s}} \chi_1(u,s_j)\, ds_j}{\int_u^{\overline{s}} \chi(u,s_j)\, ds_j}. \quad (40)$$

The assumptions that we make for the joint probability density imply that $\frac{\int_u^{\overline{s}} \chi_1(u,s_j)ds_j}{\int_u^{\overline{s}} \chi(u,s_j)ds_j}$ is bounded. Thus, $e^{-\int_s^v H(u)du}$ is strictly positive, unless

$$\begin{aligned} e^{\left[\ln\left(\int_u^{\overline{s}} \chi(u,s_j)ds_j\right)\right]_s^v} &= e^{\ln\left(\int_v^{\overline{s}} \chi(v,s_j)ds_j\right) - \ln\left(\int_s^{\overline{s}} \chi(s,s_j)ds_j\right)} \\ &= \frac{\int_v^{\overline{s}} \chi(v,s_j)\, ds_j}{\int_s^{\overline{s}} \chi(s,s_j)\, ds_j} \end{aligned}$$

is equal to zero. This is the case if and only if $\int_v^{\overline{s}} \chi(v,s_j)\, ds_j = 0$. It follows from the assumptions that we make on the joint probability distribution that this is the case if and only if $v = \overline{s}$. ■

Proof. (Proposition 3) We have $q_L = 0$ in the non-pivotal case, so it is evident that $H^*(s)$ simplifies to (19). For weakly affiliated signals, we have $\frac{d}{ds}\left(\frac{\chi(s,s_j)}{\chi(s,x)}\right) \geq 0$ if $s_j \geq x$, which ensures that the global second-order condition is satisfied when $q_L = 0$. The result now follows from Proposition 1.

By definition, we have that $\frac{dc(v,v)}{dv} = 0$ for $s < \overline{s}$ when costs are almost surely common knowledge, so it follows from (18) that

$$\begin{aligned} p(s) &= c(\underline{s},\underline{s}) + e^{-\int_s^{\overline{s}} H(u)du} \int_s^{\overline{s}} \frac{dc(v,v)}{dv} dv \\ &= c(\underline{s},\underline{s}) + e^{-\int_s^{\overline{s}} H(u)du} (\overline{p} - c(\underline{s},\underline{s})). \end{aligned}$$

The statement now follows from Lemma 4 above. ■

Uniform-price auction

The following derivations will be useful when analysing uniform-price auctions. It follows from (33) that:

$$\mathbb{E}\left[p_j - c_i\left(s_i, s_j\right)\middle| p_j \geq p_i\right] = \frac{\int_{p_j^{-1}(p_i)}^{\overline{s}} (p_j(s_j)-c_i(s_i,s_j))\chi(s_i,s_j)ds_j}{\int_{p_j^{-1}(p_i)}^{\overline{s}} \chi(s_i,s_j)ds_j}$$
$$= \frac{\int_{p_j^{-1}(p_i)}^{\overline{s}} (p_j(s_j)-c_i(s_i,s_j))\chi(s_i,s_j)ds_j}{\Pr(p_j \geq p_i|s_i) \int_{\underline{s}}^{\overline{s}} \chi(s_i,s_j)ds_j}$$
$$\frac{\partial \mathbb{E}[p_j - c_i(s_i,s_j)|p_j \geq p_i]}{\partial p_i} = \frac{p_j^{-1\prime}(p_i)\chi\left(s_i,p_j^{-1}(p_i)\right)\int_{p_j^{-1}(p_i)}^{\overline{s}}\left(p_j(s_j)-c_i(s_i,s_j)-\left(p_i-c_i\left(s_i,p_j^{-1}(p_i)\right)\right)\right)\chi(s_i,s_j)ds_j}{\left(\int_{p_j^{-1}(p_i)}^{\overline{s}}\chi(s_i,s_j)ds_j\right)^2}$$
$$= \frac{-\frac{\partial \Pr\left(p_j \geq p_i\middle|s_i\right)}{\partial p_i}\int_{p_j^{-1}(p_i)}^{\overline{s}}\left(p_j(s_i)-c_i(s_i,s_j)-\left(p_i-c_i\left(s_i,p_j^{-1}(p_i)\right)\right)\right)\chi(s_i,s_j)ds_j}{(\Pr(p_j \geq p_i|s_i))^2\int_{\underline{s}}^{\overline{s}}\chi(s_i,s_j)ds_j}. \tag{41}$$

It now follows from (33) and (41) that:

$$\frac{\partial \mathbb{E}\left[p_j - c_i\left(s_i, s_j\right)\middle| p_j \geq p_i\right]}{\partial p_i}\Pr\left(p_j \geq p_i\middle| s_i\right) \tag{42}$$
$$+\mathbb{E}\left[p_j - c_i\left(s_i, s_j\right)\middle| p_j \geq p_i\right]\frac{\partial \Pr\left(p_j \geq p_i\middle| s_i\right)}{\partial p_i}$$
$$= \frac{\partial \Pr\left(p_j \geq p_i\middle| s_i\right)}{\partial p_i}\left(p_i - c_i\left(s_i, p_j^{-1}\left(p_i\right)\right)\right).$$

Proof. (Lemma 2) We have from (20) that

$$\begin{aligned}\frac{\partial \pi_i(s_i)}{\partial p_i} = {} & \frac{\partial \mathbb{E}[p_j - c_i(s_i,s_j)|p_j \geq p_i]}{\partial p_i}\Pr\left(p_j \geq p_i\middle| s_i\right) q_H \\ & +\mathbb{E}\left[p_j - c_i\left(s_i, s_j\right)\middle| p_j \geq p_i\right]\frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} q_H \\ & +\left(1 - \frac{\partial \mathbb{E}[c_i(s_i,s_j)|p_j \leq p_i]}{\partial p_i}\right)\left(1 - \Pr\left(p_j \geq p_i\middle| s_i\right)\right) q_L \\ & -\left(p_i - \mathbb{E}\left[c_i\left(s_i, s_j\right)\middle| p_j \leq p_i\right]\right)\frac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i} q_L.\end{aligned} \tag{43}$$

It now follows from (37) and (42) that (43) can be simplified to (21). ■

Proof. (Proposition 4) We use (21) to solve for symmetric strategies, so that $p_i(s) = p_j(s) = p(s)$. We note that $p^{-1\prime}(p) = \frac{1}{p'(s)}$. Using (33) and Definition 5, the first-order condition $\frac{\partial \pi_i(s_i)}{\partial p_i} = 0$ can be written as follows:

$$\begin{aligned}\int_{\underline{s}}^{s}\chi\left(s, s_j\right) ds_j q_L - \frac{\left(p - c\left(s, s\right)\right)}{p'\left(s\right)}\chi\left(s, s\right)\left(q_H - q_L\right) &= 0 \\ p'\left(s\right) - p\hat{H}\left(s\right) &= -c\left(s, s\right)\hat{H}\left(s\right).\end{aligned}$$

The property of unaffiliated signals in (2) implies that $\frac{d}{ds}\left(\frac{\int_{\underline{s}}^{x}\chi(s,s_j)ds_j}{\chi(s,x)}\right) \geq 0$ for $x > s_j$. The statement now follows from an argument similar to that used in the proof of Proposition 1.

In the special case when costs are almost surely common knowledge, we have by definition that $\frac{dc(v,v)}{dv} = 0$ for $s < \overline{s}$, so it follows from (22) that

$$p(s) = c(\underline{s}, \underline{s}) + e^{-\int_s^{\overline{s}} \hat{H}(u)du} \int_s^{\overline{s}} \frac{dc(v,v)}{dv} dv,$$

which gives (23). ■

Proof. (Proposition 5) In the non-pivotal case, the lowest bidder sets the market price and the lowest bidder gets to produce the entire demand, which corresponds to a first-price procurement auction. In the just pivotal case, the highest bidder sets the market price and the lowest bidder gets to produce the entire demand, which corresponds to a second-price auction. Thus, the statement follows from Milgrom and Weber (1982). ■

Proof. (Proposition 6)

For independent signals we have $\chi(s, s_j) = f(s) f(s_j)$, so it follows from Definition 5 that

$$\begin{aligned} \hat{H}(u) &= \frac{(q_H - q_L) f(u)}{q_L \int_{\underline{s}}^u f(s_j) ds_j} \\ &= \frac{d}{du} \frac{(q_H - q_L) \ln \left(\int_{\underline{s}}^u f(s_j) ds_j \right)}{q_L}. \end{aligned}$$

Thus (22) can be written as in (24). Independent signals are weakly affiliated. This ensures that the sufficiency condition in Proposition 4 is satisfied.

In the special case when costs are almost surely common knowledge, we have by definition that $\frac{dc(v,v)}{dv} = 0$ for $s < \overline{s}$, so it follows from (24) that

$$p(s) = c(s, s) + (F(s))^{\frac{(q_H - q_L)}{q_L}} \int_s^{\overline{s}} \frac{dc(v,v)}{dv} dv,$$

where $F(s)$ is the marginal probability distribution. This gives (25). ■

Proof. (**Proposition 7)** We let $G(P)$ be the probability that a producer's bid is below P. This is the same as the probability that s is below $p^{-1}(P)$. Hence, it follows from (25) that

$$G(P) = \left(\frac{P - c}{\overline{p} - c} \right)^{\frac{q_L}{q_H - q_L}}.$$

From the theory of order statistics we know that

$$G^2(P) = \left(\frac{P - c}{\overline{p} - c} \right)^{\frac{2q_L}{q_H - q_L}}$$

is the probability distribution of the highest bid, which sets the price. Hence, the probability density of the market price is given by $2G(p) G'(p)$. Thus, the expected market price is given by:

$$\begin{aligned}\int_c^{\overline{p}} 2G(p)\,G'(p)\,pdp &= \left[G^2(p)\,p\right]_c^{\overline{p}} - \int_c^{\overline{p}} G^2(p)\,dp \\ &= \overline{p} - \left[\frac{(p-c)^{\frac{2q_L}{q_H-q_L}+1}}{\left(\frac{2q_L}{q_H-q_L}+1\right)(\overline{p}-c)^{\frac{2q_L}{q_H-q_L}}}\right]_c^{\overline{p}} = \overline{p} - \frac{(\overline{p}-c)(q_H-q_L)}{q_H+q_L}\end{aligned}$$

■

Proof. (Lemma 3) The demand and production capacity uncertainties are independent of the signals and cost uncertainties. Thus, when producers are pivotal with certainty, the expected profit of firm i when receiving signal s_i is:

$$\begin{aligned}\pi_i(s_i) = \mathbb{E}\left[p_j - c_i(s_i,s_j)\right| p_j \geq p_i] \Pr(p_j \geq p_i| s_i)\, q_H^P\left(1-\Pi^{NP}\right) \\ +\mathbb{E}\left[p_i(s_i) - c_i(s_i,s_j)\right| p_j \geq p_i] \Pr(p_j \geq p_i| s_i)\, q_H^{NP}\Pi^{NP} \\ +\left(p_i(s_i) - \mathbb{E}\left[c_i(s_i,s_j)\right| p_j \leq p_i]\right)\left(1-\Pr(p_j \geq p_i| s_i)\right) q_L,\end{aligned} \tag{44}$$

where

$$q_H^P = \mathbb{E}\left[\widetilde{q}_H \right| \widetilde{q}_H < D].$$

It follows from differentiation of (44) and the relations in (35), (37) and (42) that:

$$\begin{aligned}\tfrac{\partial \pi_i(s_i)}{\partial p_i} = \tfrac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i}\left(p_i - c_i\left(s_i, p_j^{-1}(p_i)\right)\right) q_H^P\left(1-\Pi^{NP}\right) \\ +\left(\Pr(p_j \geq p_i| s_i) + \left(p_i - c_i\left(s_i, p_j^{-1}(p_i)\right)\right)\tfrac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i}\right) q_H^{NP}\Pi^{NP} \\ +\tfrac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i}\left(c_i\left(s_i, p_j^{-1}(p_i)\right) - p_i\right) \\ +\left(1-\Pr(p_j \geq p_i| s_i)\right) q_L,\end{aligned}$$

so

$$\begin{aligned}\tfrac{\partial \pi_i(s_i)}{\partial p_i} = \tfrac{\partial \Pr(p_j \geq p_i|s_i)}{\partial p_i}\left(p_i - c_i\left(s_i, p_j^{-1}(p_i)\right)\right)\left(q_H^P\left(1-\Pi^{NP}\right) + q_H^{NP}\Pi^{NP} - q_L\right) \\ +\Pr(p_j \geq p_i| s_i)\, q_H^{NP}\Pi^{NP} + \left(1-\Pr(p_j \geq p_i| s_i)\right) q_L,\end{aligned}$$

which can be simplified to (27), because $q^H = q_H^P\left(1-\Pi^{NP}\right) + q_H^{NP}\Pi^{NP}$. ■

Proof. (Proposition 8) The proof is similar to the proof of Proposition 4.

■

Proof. (Proposition 9) First we note that $\chi(s,s_j) = f(s)\,f(s_j)$ for independent signals, so the inequality

$$\begin{aligned}&\frac{d}{ds}\left(\frac{\int_x^{\overline{s}} f(s)\,f(s_j)\,ds_j q_H^{NP}\Pi^{NP} + q_L\int_{\underline{s}}^x f(s)\,f(s_j)\,ds_j}{f(s)\,f(x)}\right) \\ =\; &\frac{d}{ds}\left(\frac{\int_x^{\overline{s}} f(s_j)\,ds_j q_H^{NP}\Pi^{NP} + q_L\int_{\underline{s}}^x f(s_j)\,ds_j}{f(x)}\right) = 0 \geq 0\end{aligned}$$

is satisfied. Thus, it follows from Proposition 8 that the global second-order condition is satisfied. Moreover, for independent signals, we have from Definition 6 that

$$\begin{aligned}\tilde{H}(s) &= \frac{f(s)(q_H - q_L)}{\int_s^{\overline{s}} f(s_j)\, ds_j q_H^{NP} \Pi^{NP} + \int_{\underline{s}}^{s} f(s_j)\, ds_j q_L} \qquad (45)\\ &= -\frac{d}{ds} \ln\left(\int_s^{\overline{s}} f(s_j)\, ds_j q_H^{NP} \Pi^{NP} + \int_{\underline{s}}^{s} f(s_j)\, ds_j q_L\right) \frac{(q_H - q_L)}{q_H^{NP}\Pi^{NP} - q_L},\end{aligned}$$

Thus, (28) can be written as in (30).

In case costs are almost surely common knowledge, so that $\frac{dc(v,v)}{dv} \searrow 0$ for $v < \overline{s}$, (30) can be simplified to (31) as follows:

$$\begin{aligned}p(s) &= c(\underline{s}, \underline{s}) + \left(\frac{q_L}{(1 - F(s))\, q_H^{NP}\Pi^{NP} + F(s)\, q_L}\right)^{\frac{(q_H - q_L)}{q_H^{NP}\Pi^{NP} - q_L}} \int_s^{\overline{s}} \frac{dc(v,v)}{dv} dv\\ &= c(\underline{s}, \underline{s}) + \left(\frac{q_L}{((1 - F(s))\, q_H^{NP}\Pi^{NP} + F(s)\, q_L)}\right)^{\frac{(q_H - q_L)}{q_H^{NP}\Pi^{NP} - q_L}} (\overline{p} - c(\underline{s}, \underline{s})),\end{aligned}$$

where $F(s)$ is the marginal probability distribution. ■

Ranking of auction formats

Proof. (Proposition 10)

When costs are almost surely common knowledge, in a uniform-price auction we have that:

$$\begin{aligned}\pi_i(s) = &\ \mathbb{E}[p_j - c(\underline{s}, \underline{s}) |\, p_j \geq p_i] \Pr(p_j \geq p_i |\, s)\, q_H^P \left(1 - \Pi^{NP}\right)\\ &+ (p_i(s) - c(\underline{s}, \underline{s})) \Pr(p_j \geq p_i |\, s)\, q_H^{NP}\Pi^{NP}\\ &+ (p_i(s) - c(\underline{s}, \underline{s}))(1 - \Pr(p_j \geq p_i |\, s))\, q_L,\end{aligned}$$

Independent cost information yields:

$$\begin{aligned}\pi_i(s) = &\ \mathbb{E}[p_j - c(\underline{s}, \underline{s}) |\, p_j \geq p_i](1 - F(s))\, q_H^P \left(1 - \Pi^{NP}\right)\\ &+ (p_i(s) - c(\underline{s}, \underline{s}))(1 - F(s))\, q_H^{NP}\Pi^{NP}\\ &+ (p_i(s_i) - c(\underline{s}, \underline{s}))\, F(s)\, q_L.\end{aligned} \qquad (46)$$

We have from (31) that:

$$\begin{aligned}&\frac{\mathbb{E}[p_j - c(\underline{s}, \underline{s}) |\, p_j \geq p_i](1 - F(s))}{(\overline{p} - c(\underline{s}, \underline{s}))}\\ &= \int_s^{\overline{s}} \left(\frac{q_L}{((1 - F(s_j))\, q_H^{NP}\Pi^{NP} + F(s_j)\, q_L)}\right)^{\frac{(q_H - q_L)}{q_H^{NP}\Pi^{NP} - q_L}} \chi ds_j\\ &= \int_s^{\overline{s}} \left(\frac{((1 - F(s_j))\, q_H^{NP}\Pi^{NP} + F(s_j)\, q_L)}{q_L}\right)^{\frac{-(q_H - q_L)}{q_H^{NP}\Pi^{NP} - q_L}} \chi ds_j\end{aligned}$$

We make the substitution $u = F(s_j)$ and use that $\chi = F'(s_j)$, so

$$
\begin{aligned}
&= \int_F^1 \left(\frac{\left((1-u)\, q_H^{NP}\Pi^{NP} + uq_L\right)}{q_L} \right)^{\frac{-(q_H - q_L)}{q_H^{NP}\Pi^{NP} - q_L}} du \\
&= \left[\frac{\left(\frac{\left((1-u)q_H^{NP}\Pi^{NP}+uq_L\right)}{q_L} \right)^{1-\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}}}{\left(1 - \frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}\right)\left(\frac{\left(-q_H^{NP}\Pi^{NP}+q_L\right)}{q_L}\right)} \right]_F^1 \\
&= \frac{1}{\left(1 - \frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}\right)\left(\frac{-q_H^{NP}\Pi^{NP}+q_L}{q_L}\right)} - \frac{\left(\frac{\left((1-F)q_H^{NP}\Pi^{NP}+Fq_L\right)}{q_L} \right)^{1-\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}}}{\left(1 - \frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}\right)\left(\frac{-q_H^{NP}\Pi^{NP}+q_L}{q_L}\right)} \\
&= \frac{q_L}{\left(q_H - q_H^{NP}\Pi^{NP}\right)} \left(1 - \left(\frac{\left((1-F)\, q_H^{NP}\Pi^{NP} + Fq_L\right)}{q_L} \right)^{1-\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}} \right) \\
&= \frac{q_L}{q_H^P\left(1 - \Pi^{NP}\right)} \left(1 - \left(\frac{\left((1-F)\, q_H^{NP}\Pi^{NP} + Fq_L\right)}{q_L} \right)^{1-\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}} \right),
\end{aligned}
$$

because $q_H = q_H^{NP}\Pi^{NP} + q_H^P\left(1 - \Pi^{NP}\right)$. We substitute this result and (31) into (44), which yields:

$$
\begin{aligned}
\tfrac{\pi_i(s)}{\overline{p}-c(\underline{s},\underline{s})} &= q_L \left(1 - \left(\frac{\left((1-F)q_H^{NP}\Pi^{NP}+Fq_L\right)}{q_L} \right)^{1-\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}} \right) \\
&+ \left(\frac{q_L}{\left((1-F(s))q_H^{NP}\Pi^{NP}+F(s)q_L\right)} \right)^{\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}} (1 - F(s))\, q_H^{NP}\Pi^{NP} \\
&+ \left(\frac{q_L}{\left((1-F(s))q_H^{NP}\Pi^{NP}+F(s)q_L\right)} \right)^{\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}} F(s)\, q_L \\
&= q_L \left(1 - \left(\frac{\left((1-F)q_H^{NP}\Pi^{NP}+Fq_L\right)}{q_L} \right)^{1-\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}} \right) \\
&+ q_L^{\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}} \left(\left((1-F(s))\, q_H^{NP}\Pi^{NP} + F(s)\, q_L\right)\right)^{1-\frac{(q_H-q_L)}{q_H^{NP}\Pi^{NP}-q_L}} \\
&= q_L.
\end{aligned}
$$

because $q_H = q_H^{NP}\Pi^{NP} + q_H^P\left(1 - \Pi^{NP}\right)$.

Next we derive the same result for the discriminatory auction. It follows from (7) and (16) that

$$
\begin{aligned}
\pi(s) &= (p(s) - c(\underline{s},\underline{s}))\,(1 - F(s))\, q_H + (p(s) - c(\underline{s},\underline{s}))\, F(s)\, q_L \\
&= q_L\left(\overline{p} - c(\underline{s},\underline{s})\right).
\end{aligned}
$$

■

Proof. (Proposition 11) It follows from (44) that the expected revenue of a producer in a uniform price auction after observing the signal s is:

$$R_u(s) = \frac{\int_s^{\overline{s}} p(s_j)\chi(s,s_j)ds_j q_H^P\left(1-\Pi^{NP}\right)+\int_s^{\overline{s}} p(s)\chi(s,s_j)ds_j q_H^{NP}\Pi^{NP}+\int_{\underline{s}}^s p(s)\chi(s,s_j)ds_j q_L}{\int_{\underline{s}}^{\overline{s}}\chi(s,s_j)ds_j}$$
$$= \frac{\int_s^{\overline{s}}\left(c(s_j,s_j)+\int_{s_j}^{\overline{s}}\frac{dc(v,v)}{dv}e^{-\int_{s_j}^v \widetilde{H}(u)du}dv\right)\chi(s,s_j)ds_j q_H^P\left(1-\Pi^{NP}\right)}{\int_{\underline{s}}^{\overline{s}}\chi(s,s_j)ds_j}$$
$$+\frac{\int_s^{\overline{s}}\left(c(s,s)+\int_s^{\overline{s}}\frac{dc(v,v)}{dv}e^{-\int_s^v \widetilde{H}(u)du}dv\right)\chi(s,s_j)ds q_H^{NP}\Pi^{NP}}{\int_{\underline{s}}^{\overline{s}}\chi(s,s_j)ds_j}$$
$$+\frac{\int_{\underline{s}}^s\left(c(s,s)+\int_s^{\overline{s}}\frac{dc(v,v)}{dv}e^{-\int_s^v \widetilde{H}(u)du}dv\right)\chi(s,s_j)ds_j q_L}{\int_{\underline{s}}^{\overline{s}}\chi(s,s_j)ds_j}. \tag{47}$$

Let

$$\Theta_u(s) = \int_s^{\overline{s}}\int_{s_j}^{\overline{s}}\frac{dc(v,v)}{dv}e^{-\int_{s_j}^v \widetilde{H}(u)du}dv\chi(s,s_j)\,ds_j q_H^P\left(1-\Pi^{NP}\right)$$
$$+\frac{\int_s^{\overline{s}}\int_s^{\overline{s}}\frac{dc(v,v)}{dv}e^{-\int_s^v \widetilde{H}(u)du}dv\;\chi(s,s_j)ds_j q_H^{NP}\Pi^{NP}}{\int_{\underline{s}}^{\overline{s}}\chi(s,s_j)ds_j}$$
$$+\int_{\underline{s}}^s\int_s^{\overline{s}}\frac{dc(v,v)}{dv}\left(\frac{\int_{\underline{s}}^s\chi(s,u)du}{\int_{\underline{s}}^v\chi(v,u)du}\right)^{\frac{(q_H-q_L)}{q_L}}dv\chi(s,s_j)\,ds_j q_L.$$

We can change the order of integration as follows:

$$\begin{aligned}\Theta_u(s) &= \int_s^{\overline{s}}\frac{dc(v,v)}{dv}\int_s^v e^{-\int_{s_j}^v \widetilde{H}(u)du}\chi(s,s_j)\,ds_j dv q_H^P\left(1-\Pi^{NP}\right)\\ &+\int_s^{\overline{s}}\frac{dc(v,v)}{dv}\int_s^{\overline{s}} e^{-\int_s^v \widetilde{H}(u)du}\chi(s,s_j)\,ds_j dv\; q_H^{NP}\Pi^{NP}\\ &+\int_s^{\overline{s}}\frac{dc(v,v)}{dv}\int_{\underline{s}}^s e^{-\int_s^v \widetilde{H}(u)du}\chi(s,s_j)\,ds_j dv q_L.\end{aligned}$$

Assume now that $\frac{dc(v,v)}{dv}$ is zero for v below $w \geq s$. In this case, we have:

$$\begin{aligned}\Theta_u(s) &= \int_w^{\overline{s}}\frac{dc(v,v)}{dv}\int_s^v e^{-\int_{s_j}^v \widetilde{H}(u)du}\chi(s,s_j)\,ds_j dv q_H^P\left(1-\Pi^{NP}\right)\\ &+\int_w^{\overline{s}}\frac{dc(v,v)}{dv}\int_s^{\overline{s}} e^{-\int_s^v \widetilde{H}(u)du}\chi(s,s_j)\,ds_j dv\; q_H^{NP}\Pi^{NP}\\ &+\int_w^{\overline{s}}\frac{dc(v,v)}{dv}\int_{\underline{s}}^s e^{-\int_s^v \widetilde{H}(u)du}\chi(s,s_j)\,ds_j dv q_L,\end{aligned}$$

so that

$$\begin{aligned}\frac{d\Theta_u(s)}{dw} &= -\frac{dc(w,w)}{dw}\int_s^w e^{-\int_{s_j}^w \widetilde{H}(u)du}\chi(s,s_j)\,ds_j q_H^P\left(1-\Pi^{NP}\right)\\ &-\frac{dc(w,w)}{dw}e^{-\int_s^w \widetilde{H}(u)du}\left(\int_{\underline{s}}^s\chi(s,s_j)\,ds_j q_L+\int_s^{\overline{s}}\chi(s,s_j)\,ds_j q_H^{NP}\Pi^{NP}\right),\end{aligned}$$

if $w \geq s$. Obviously, $\frac{d\Theta_u(s)}{dw} = 0$ if $w < s$. Now, we use that signals are independent, so $\chi(s, s_j) = f(s) f(s_j)$ and it follows from (45) that

$$\begin{aligned}
\widetilde{H}(s) &= -\frac{d}{ds} \ln\left((1 - F(s)) q_H^{NP} \Pi^{NP} + F(s) q_L\right) \frac{(q_H - q_L)}{q_H^{NP} \Pi^{NP} - q_L} \\
\int_s^w \widetilde{H}(u)\, du &= \left[\frac{(q_H - q_L) \ln\left((1 - F(s)) q_H^{NP} \Pi^{NP} + F(s) q_L\right)}{q_L - q_H^{NP} \Pi^{NP}}\right]_s^w \\
&= \frac{(q_H - q_L) \ln\left(\frac{(1-F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L}{(1-F(s)) q_H^{NP} \Pi^{NP} + F(s) q_L}\right)}{q_L - q_H^{NP} \Pi^{NP}} \\
e^{-\int_s^w \widetilde{H}(u) du} &= \left(\frac{(1 - F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L}{(1 - F(s)) q_H^{NP} \Pi^{NP} + F(s) q_L}\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}}.
\end{aligned}$$

Next, we use the above expressions and that $\chi(s, s_j) = f(s) f(s_j)$, so

$$\begin{aligned}
\frac{d\Theta_u(s)}{dw} = &-\frac{dc(w,w)}{dw} \int_s^w \left(\frac{(1-F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L}{(1-F(s_j)) q_H^{NP} \Pi^{NP} + F(s_j) q_L}\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} f(s) f(s_j)\, ds_j q_H^P \left(1 - \Pi^{NP}\right) \\
&-\frac{dc(w,w)}{dw} \int_s^{\overline{s}} \left(\frac{(1-F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L}{(1-F(s)) q_H^{NP} \Pi^{NP} + F(s) q_L}\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} f(s) f(s_j)\, ds_j q_H^{NP} \Pi^{NP} \\
&-\frac{dc(w,w)}{dw} \int_{\underline{s}}^s \left(\frac{(1-F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L}{(1-F(s)) q_H^{NP} \Pi^{NP} + F(s) q_L}\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} f(s) f(s_j)\, ds_j q_L.
\end{aligned} \tag{48}$$

Thus, $\frac{d\Theta_u(s)}{dw}$ can be simplified as follows:

$$\begin{aligned}
\frac{d\Theta_u(s)}{dw} = &-\frac{dc(w,w)}{dw} \int_{F(s)}^{F(w)} \left(\frac{(1-F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L}{(1-F) q_H^{NP} \Pi^{NP} + F q_L}\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} f(s)\, dF q_H^P \left(1 - \Pi^{NP}\right) \\
&-\frac{dc(w,w)}{dw} \left(\frac{(1-F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L}{(1-F(s)) q_H^{NP} \Pi^{NP} + F(s) q_L}\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} f(s) (1 - F(s)) q_H^{NP} \Pi^{NP} \\
&-\frac{dc(w,w)}{dw} \left(\frac{(1-F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L}{(1-F(s)) q_H^{NP} \Pi^{NP} + F(s) q_L}\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} f(s) F(s) q_L.
\end{aligned}$$

The first integral can be solved as follows:

$$\begin{aligned}
&\int_{F(s)}^{F(w)} \left(\frac{(1 - F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L}{(1 - F) q_H^{NP} \Pi^{NP} + F q_L}\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} dF \\
= &\left((1 - F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} \int_{F(s)}^{F(w)} \left((1 - F) q_H^{NP} \Pi^{NP} + F q_L\right)^{\frac{(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} dF \\
= &\left((1 - F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} \left[\frac{\left((1 - F) q_H^{NP} \Pi^{NP} + F q_L\right)^{\frac{(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}} + 1}}{\left(q_H - q_H^{NP} \Pi^{NP}\right)}\right]_{F(s)}^{F(w)} \\
= &\left((1 - F(w)) q_H^{NP} \Pi^{NP} + F(w) q_L\right)^{\frac{-(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}}} \left[\frac{\left((1 - F) q_H^{NP} \Pi^{NP} + F q_L\right)^{\frac{(q_H - q_L)}{q_L - q_H^{NP} \Pi^{NP}} + 1}}{q_H^P \left(1 - \Pi^{NP}\right)}\right]_{F(s)}^{F(w)},
\end{aligned}$$

because $q_H = q_H^{NP}\Pi^{NP} + q_H^P\left(1-\Pi^{NP}\right)$. Using this integral, we can simplify $\frac{d\Theta_u(s)}{dw}$ as follows:

$$\begin{aligned}\frac{d\Theta_u(s)}{dw} &= -\frac{dc(w,w)}{dw}\left((1-F(w))\,q_H^{NP}\Pi^{NP}+F(w)\,q_L\right)^{\frac{-(q_H-q_L)}{q_L-q_H^{NP}\Pi^{NP}}} f(s)\\ &\quad\left[\left((1-F)\,q_H^{NP}\Pi^{NP}+Fq_L\right)^{\frac{(q_H-q_L)}{q_L-q_H^{NP}\Pi^{NP}}+1}\right]_{F(s)}^{F(w)}\\ &-\frac{dc(w,w)}{dw}\left(\frac{(1-F(w))q_H^{NP}\Pi^{NP}+F(w)q_L}{(1-F(s))q_H^{NP}\Pi^{NP}+F(s)q_L}\right)^{\frac{-(q_H-q_L)}{q_L-q_H^{NP}\Pi^{NP}}} f(s)\,(1-F(s))\,q_H^{NP}\Pi^{NP}\\ &\quad-\frac{dc(w,w)}{dw}\left(\frac{(1-F(w))q_H^{NP}\Pi^{NP}+F(w)q_L}{(1-F(s))q_H^{NP}\Pi^{NP}+F(s)q_L}\right)^{\frac{-(q_H-q_L)}{q_L-q_H^{NP}\Pi^{NP}}} f(s)\,F(s)\,q_L\\ &\quad= -\frac{dc(w,w)}{dw}\left((1-F(w))\,q_H^{NP}\Pi^{NP}+F(w)\,q_L\right) f(s)\,.\end{aligned} \tag{49}$$

Next, we make the corresponding calculation for the discriminatory auction. The expected revenue in a discriminatory auction is:

$$\begin{aligned}R_d(s) &= \frac{\int_s^{\overline{s}} p(s)\chi(s,s_j)ds_j q_H + \int_{\underline{s}}^{s} p(s)\chi(s,s_j)ds_j q_L}{\int_{\underline{s}}^{\overline{s}}\chi(s,s_j)ds_j}\\ &= \frac{\int_s^{\overline{s}} c(s,s)\chi(s,s_j)q_H ds_j + \int_{\underline{s}}^{s} c(s,s)\chi(s,s_j)q_L ds_j + \Theta_d(s)}{\int_{\underline{s}}^{\overline{s}}\chi(s,s_j)ds_j}.\end{aligned} \tag{50}$$

We have independent signals, so it follows from Proposition 2 that

$$\begin{aligned}\Theta_d(s) &= \int_s^{\overline{s}}\int_s^{\overline{s}}\frac{dc(v,v)}{dv}\left(\frac{(1-F(v))\,q_H+F(v)\,q_L}{(1-F(s))\,q_H+F(s)\,q_L}\right)dvf(s)\,f(s_j)\,ds_jq_H\\ &+\int_{\underline{s}}^{s}\int_s^{\overline{s}}\frac{dc(v,v)}{dv}\left(\frac{(1-F(v))\,q_H+F(v)\,q_L}{(1-F(s))\,q_H+F(s)\,q_L}\right)dvf(s)\,f(s_j)\,ds_jq_L.\end{aligned}$$

As for the uniform-price auction, we change the order of integration:

$$\begin{aligned}\Theta_d(s) &= \int_s^{\overline{s}}\frac{dc(v,v)}{dv}\int_s^{\overline{s}}\left(\frac{(1-F(v))\,q_H+F(v)\,q_L}{(1-F(s))\,q_H+F(s)\,q_L}\right)f(s)\,f(s_j)\,ds_jdvq_H\\ &+\int_s^{\overline{s}}\frac{dc(v,v)}{dv}\int_{\underline{s}}^{s}\left(\frac{(1-F(v))\,q_H+F(v)\,q_L}{(1-F(s))\,q_H+F(s)\,q_L}\right)f(s)\,f(s_j)\,ds_jdvq_L.\end{aligned}$$

Next, we assume that $\frac{dc(v,v)}{dv}$ is zero for v below $w \geq s$. In this case, we have:

$$\begin{aligned}\Theta_d(s) &= \int_w^{\overline{s}}\frac{dc(v,v)}{dv}\int_s^{\overline{s}}\left(\frac{(1-F(v))\,q_H+F(v)\,q_L}{(1-F(s))\,q_H+F(s)\,q_L}\right)f(s)\,f(s_j)\,ds_jdvq_H\\ &+\int_w^{\overline{s}}\frac{dc(v,v)}{dv}\int_{\underline{s}}^{s}\left(\frac{(1-F(v))\,q_H+F(v)\,q_L}{(1-F(s))\,q_H+F(s)\,q_L}\right)f(s)\,f(s_j)\,ds_jdvq_L,\end{aligned}$$

so that

$$\begin{aligned}\frac{d\Theta_d(s)}{dw} &= -\frac{dc(w,w)}{dw}\int_s^{\overline{s}}\left(\frac{(1-F(w))\,q_H+F(w)\,q_L}{(1-F(s))\,q_H+F(s)\,q_L}\right)f(s)\,f(s_j)\,ds_jq_H\\ &\quad-\frac{dc(w,w)}{dw}\int_{\underline{s}}^{s}\left(\frac{(1-F(w))\,q_H+F(w)\,q_L}{(1-F(s))\,q_H+F(s)\,q_L}\right)f(s)\,f(s_j)\,ds_jq_L,\end{aligned}$$

if $w \geq s$. Obviously, $\frac{d\Theta_d(s)}{dw} = 0$ if $w < s$.

$$\begin{aligned} \frac{d\Theta_d(s)}{dw} &= -\frac{dc(w,w)}{dw}\left(\frac{(1-F(w))q_H + F(w)q_L}{(1-F(s))q_H + F(s)q_L}\right) f(s)(1-F(s))\, q_H \\ &\quad -\frac{dc(w,w)}{dw}\left(\frac{(1-F(w))q_H + F(w)q_L}{(1-F(s))q_H + F(s)q_L}\right) f(s) F(s)\, q_L. \end{aligned}$$

if $w \geq s$. Next we use that χ is a constant.

$$\begin{aligned} \frac{d\Theta_d(s)}{dw} &= -\frac{dc(w,w)}{dw}\left(\frac{(1-F(w))q_H + F(w)q_L}{(1-F(s))q_H + F(s)q_L}\right) f(s)\left((1-F(s))\, q_H + F(s)\, q_L\right) \\ &= -\frac{dc(w,w)}{dw}\left((1-F(w))\, q_H + F(w)\, q_L\right) f(s)\,. \end{aligned} \tag{51}$$

Thus, it follows from (49) and (51) that $\frac{d\Theta_d(s)}{dw} \leq \frac{d\Theta_u(s)}{dw}$, which also implies that $\frac{dR_d(s)}{dw} \leq \frac{dR_u(s)}{dw}$. We have from Proposition 11 that $R_d \geq R_u$ for $w = \overline{s}$ when the signals are independent. Thus, we can conclude that $R_d \geq R_u$ for $w \leq \overline{s}$ and independent signals. By comparing (48) and (51), we can also deduce that R_u decreases, and accordingly $R_d - R_u$ increases, when $q_H^{NP}\Pi^{NP}$ decreases for a fixed q_H. ■